

IPhO 1985

Theory problems and solutions · Portorož, Yugoslavia · 3 theory problems with complete solutions

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Problem 1: Aerial array for a radio link

A young radio amateur maintains a radio link with two girls living in two towns. He positions an aerial array such that when the girl living in town A receives a maximum signal, the girl living in town B receives no signal and vice versa. The array is built from two vertical rod aerials transmitting with equal intensities uniformly in all directions in the horizontal plane.

- Find the parameters of the array, i.e. the distance between the rods, its orientation and the phase shift between the electrical signals supplied to the rods, such that the distance between the rods is minimum.
- Find the numerical solution if the boy has a radio station transmitting at 27 MHz and builds up the aerial array at Portorož. Using the map he has found that the angles between the north and the direction of A (Koper) and of B (small town of Buje in Istria) are 72° and 157° , respectively.

Official English text of the problem.

Answers

- a)** Let α be the angular separation of the towns as viewed from the array, and let λ be the wavelength. The minimum spacing is

$$d_{\min} = \frac{\lambda}{4 \sin(\alpha/2)}.$$

The line joining the rods must be perpendicular to the internal bisector of the two town directions. Label the rods so that rod 2 lies toward $\hat{\mathbf{n}}_A - \hat{\mathbf{n}}_B$ from rod 1, where the unit vectors point toward the towns. Rod 2 must **lag rod 1 by 90°** for A to receive a maximum and B a null; it must **lead by 90°** for the reverse.

- **b)** The angular separation is 85° , giving

$$d_{\min} \approx 4.1 \text{ m}.$$

The rod-to-rod line has bearings

$$24.5^\circ \text{ and } 204.5^\circ$$

measured clockwise from north. The northeastern rod must lag the southwestern rod by 90° for maximum reception at Koper and zero reception at Buje; reversing the phase relation exchanges the two.

Solution

a)

The signals from the rods are coherent, have the same frequency, and have matching vertical polarizations. We use the far-field approximation: the towns are much farther from the array than the rod spacing, so the waves reaching either town have equal amplitudes and effectively parallel propagation directions.

Let the horizontal displacement from rod 1 to rod 2 be $\mathbf{d}$, with magnitude d . Define the electrical phase difference by

$$\delta = \phi_2 - \phi_1,$$

where the source oscillations are proportional to $\cos(\omega t + \phi_j)$. Thus positive δ means that rod 2 leads rod 1.

For a distant receiver in direction $\hat{\mathbf{n}}$, the path lengths satisfy

$$r_2 - r_1 \simeq -\hat{\mathbf{n}} \cdot \mathbf{d}.$$

Consequently, the phase difference between the received waves is

$$\Delta(\hat{\mathbf{n}}) = \delta - k(r_2 - r_1) = \delta + k\hat{\mathbf{n}} \cdot \mathbf{d}, \quad k = \frac{2\pi}{\lambda}.$$

If either rod alone produces intensity I_0 at that receiver, superposition gives

$$I = 2I_0(1 + \cos \Delta) = 4I_0 \cos^2\left(\frac{\Delta}{2}\right).$$

A maximum therefore requires $\Delta = 2m\pi$, whereas a null requires $\Delta = (2n + 1)\pi$.

For a maximum at A and a null at B,

$$\begin{aligned} \delta + k\hat{\mathbf{n}}_A \cdot \mathbf{d} &= 2m\pi, \\ \delta + k\hat{\mathbf{n}}_B \cdot \mathbf{d} &= (2n + 1)\pi. \end{aligned}$$

Subtracting eliminates the adjustable electrical phase:

$$k(\hat{\mathbf{n}}_A - \hat{\mathbf{n}}_B) \cdot \mathbf{d} = \text{an odd multiple of } \pi.$$

Thus

$$|(\hat{\mathbf{n}}_A - \hat{\mathbf{n}}_B) \cdot \mathbf{d}| \geq \frac{\lambda}{2}.$$

Since the angle between the two unit vectors is α ,

$$|\hat{\mathbf{n}}_A - \hat{\mathbf{n}}_B| = \sqrt{2 - 2 \cos \alpha} = 2 \sin \frac{\alpha}{2}.$$

The magnitude of a scalar product cannot exceed the product of the vector magnitudes, so

$$2d \sin \frac{\alpha}{2} \geq \frac{\lambda}{2}.$$

It follows that

$$d_{\min} = \frac{\lambda}{4 \sin(\alpha/2)}.$$

Equality is attained when $\mathbf{d}$ is parallel to $\hat{\mathbf{n}}_A - \hat{\mathbf{n}}_B$. Moreover,

$$(\hat{\mathbf{n}}_A - \hat{\mathbf{n}}_B) \cdot (\hat{\mathbf{n}}_A + \hat{\mathbf{n}}_B) = 0.$$

Because the sum points along the internal angular bisector, the required rod-to-rod line is perpendicular to that bisector.

Choose the direction from rod 1 to rod 2 along $\hat{\mathbf{n}}_A - \hat{\mathbf{n}}_B$. At the minimum spacing,

$$\hat{\mathbf{n}}_A \cdot \mathbf{d} = d_{\min} \sin \frac{\alpha}{2} = \frac{\lambda}{4}, \quad \hat{\mathbf{n}}_B \cdot \mathbf{d} = -\frac{\lambda}{4}.$$

Hence the received phase differences are

$$\Delta_A = \delta + \frac{\pi}{2}, \quad \Delta_B = \delta - \frac{\pi}{2}.$$

For maximum reception at A, choose

$$\delta = -\frac{\pi}{2} \pmod{2\pi}.$$

Then $\Delta_A = 0$ and $\Delta_B = -\pi$, as required.

For maximum reception at B, choose

$$\delta = +\frac{\pi}{2} \pmod{2\pi}.$$

Then $\Delta_B = 0$ and $\Delta_A = \pi$. Switching between the two operating modes therefore requires a 180° change in the relative electrical phase.

b)

The angular separation of the towns is

$$\alpha = 157^\circ - 72^\circ = 85^\circ.$$

Using $c \simeq 3.00 \times 10^8 \text{ m s}^{-1}$, the wavelength is

$$\lambda = \frac{c}{f} = \frac{3.00 \times 10^8}{27 \times 10^6} \simeq 11.1 \text{ m}.$$

Therefore,

$$d_{\min} = \frac{11.1}{4 \sin 42.5^\circ} \text{ m} \simeq 4.11 \text{ m},$$

or approximately

$$d_{\min} = 4.1 \text{ m}.$$

The internal bisector has bearing

$$\frac{72^\circ + 157^\circ}{2} = 114.5^\circ.$$

The perpendicular rod-to-rod line consequently has bearing

$$114.5^\circ - 90^\circ = 24.5^\circ,$$

with the opposite direction at 204.5° .

Label the southwestern rod 1 and the northeastern rod 2. The projections toward the towns are

$$\hat{\mathbf{n}}_A \cdot \mathbf{d} = d \cos(72^\circ - 24.5^\circ) = d \sin 42.5^\circ = \frac{\lambda}{4},$$

$$\hat{\mathbf{n}}_B \cdot \mathbf{d} = d \cos(157^\circ - 24.5^\circ) = -\frac{\lambda}{4}.$$

Thus the phase settings are:

Desired reception	Phase of northeastern rod relative to southwestern rod
Koper maximum; Buje zero	Lags by 90°
Buje maximum; Koper zero	Leads by 90°

The solution was checked against the official solution.

Problem 2: Hall effect in an InSb bar

In a long bar having the shape of a rectangular parallelepiped with sides a , b , and c ($a \gg b \gg c$), made from the semiconductor InSb flows a current I parallel to the edge a . The bar is in an external magnetic field B which is parallel to the edge c . The magnetic field produced by the current I can be neglected. The current carriers are electrons. The average velocity of electrons in a semiconductor in the presence of an electric field only is $v = \mu E$, where μ is called mobility. If the magnetic field is also present, the electric field is no longer parallel to the current. This phenomenon is known as the Hall effect.

- Determine what the magnitude and the direction of the electric field in the bar is, to yield the current described above.
- Calculate the difference of the electric potential between the opposite points on the surfaces of the bar in the direction of the edge b .
- Find the analytic expression for the DC component of the electric potential difference in case b) if the current and the magnetic field are alternating (AC); $I = I_0 \sin \omega t$ and $B = B_0 \sin(\omega t + \delta)$.
- Design and explain an electric circuit which would make possible, by exploiting the result c), to measure the power consumption of an electric apparatus connected with the AC network.

Data: The electron mobility in InSb is $7.8 \text{ m}^2\text{T/Vs}$, the electron concentration in InSb is $2.5 \cdot 10^{22} \text{ m}^{-3}$, $I = 1.0 \text{ A}$, $B = 0.10 \text{ T}$, $b = 1.0 \text{ cm}$, $c = 1.0 \text{ mm}$, $e_0 = -1.6 \cdot 10^{-19} \text{ As}$.

Official English text of the problem.

Answers

Choose right-handed axes with $+\hat{\mathbf{x}}$ along the conventional current and $+\hat{\mathbf{z}}$ along the magnetic field. Let $e = -e_0 > 0$, let n be the electron concentration, and define the Hall voltage as

$$U_H = V(y = +b/2) - V(y = -b/2).$$

- **a)** The electric field is

$$\mathbf{E} = \frac{I}{necb\mu}\hat{\mathbf{x}} - \frac{IB}{necb}\hat{\mathbf{y}} = (3.2\hat{\mathbf{x}} - 2.5\hat{\mathbf{y}}) \text{ V m}^{-1}.$$

Its magnitude is 4.1 V m^{-1} . It is perpendicular to $\mathbf{B}$ and tilted through

$$\theta = \tan^{-1}(\mu B) = 38^\circ$$

from the current direction toward $-\hat{\mathbf{y}}$, the direction of $\mathbf{j} \times \mathbf{B}$.

- **b)**

$$U_H = \frac{IB}{nec} = 2.5 \times 10^{-2} \text{ V} = 25 \text{ mV}.$$

The $+y$ face is at higher potential.

- **c)**

$$U_{H,\text{DC}} = \frac{I_0 B_0}{2nec} \cos \delta.$$

- **d)** Use a series current coil to produce $B(t) = k_I i_L(t)$, and drive the Hall bar through a resistive voltage-sensing branch so that $I_H(t) = k_U u(t)$. An averaging voltmeter across the b -faces then reads

$$\overline{U_H} = \frac{k_U k_I}{nec} P, \quad P = U_{\text{rms}} I_{L,\text{rms}} \cos \varphi.$$

Calibrate its scale directly in watts.

Solution

We consider the uniform central region of the long bar, neglecting contact effects and the magnetic field generated by its current. The transverse faces are open-circuit, so there is no steady transverse current.

For numerical calculations, the mobility is interpreted as $7.8 \text{ m}^2/(\text{V s})$, as required by $v = \mu E$; the unit $\text{m}^2\text{T/Vs}$ given in the data is dimensionally inconsistent with this relation.

a) Magnitude and direction of the electric field

The current density and electron drift velocity are related by

$$\mathbf{j} = -ne\mathbf{v}_e.$$

Since the cross-sectional area is bc ,

$$\mathbf{j} = \frac{I}{bc}\hat{\mathbf{x}}, \quad \mathbf{v}_e = -v_d\hat{\mathbf{x}}, \quad v_d = \frac{I}{nebc}.$$

The given mobility law relates speed to field magnitude. In vector form, electrons drift opposite to the electric field when no magnetic field is present.

In the usual isotropic mobility model, collisions exert a mean resisting force $-e\mathbf{v}_e/\mu$. The steady force balance in both fields is therefore

$$0 = -e(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \frac{e}{\mu}\mathbf{v}_e.$$

Consequently,

$$\mathbf{E} = -\frac{\mathbf{v}_e}{\mu} - \mathbf{v}_e \times \mathbf{B}.$$

With $\mathbf{B} = B\hat{\mathbf{z}}$,

$$\mathbf{v}_e \times \mathbf{B} = (-v_d\hat{\mathbf{x}}) \times (B\hat{\mathbf{z}}) = v_d B\hat{\mathbf{y}},$$

so

$$\boxed{\mathbf{E} = \frac{v_d}{\mu}\hat{\mathbf{x}} - v_d B\hat{\mathbf{y}}.}$$

The longitudinal component maintains the drift against collisions. The transverse component balances the magnetic force. In particular, the magnetic force initially pushes electrons toward the $-y$ face; their accumulation produces the required field in the $-y$ direction.

Numerically,

$$bc = (1.0 \times 10^{-2})(1.0 \times 10^{-3}) = 1.0 \times 10^{-5} \text{ m}^2, \\ v_d = \frac{1.0}{(2.5 \times 10^{22})(1.6 \times 10^{-19})(1.0 \times 10^{-5})} = 25 \text{ m s}^{-1}.$$

Thus

$$E_x = \frac{25}{7.8} = 3.2 \text{ V m}^{-1}, \quad E_y = -25(0.10) = -2.5 \text{ V m}^{-1}, \quad E_z = 0.$$

The magnitude is

$$E = \sqrt{E_x^2 + E_y^2} = \frac{I}{nebc\mu} \sqrt{1 + (\mu B)^2} = \boxed{4.1 \text{ V m}^{-1}}.$$

The angle from $+\hat{\mathbf{x}}$ toward $-\hat{\mathbf{y}}$ satisfies

$$\tan \theta = \frac{|E_y|}{E_x} = \mu B = 0.78,$$

giving

$$\boxed{\theta = 38^\circ}.$$

b) Transverse potential difference

Choose opposite points on the two b -faces at the same longitudinal coordinate. Since $\mathbf{E} = -\nabla V$,

$$U_H = - \int_{-b/2}^{b/2} E_y dy = -E_y b = v_d B b.$$

Substituting the drift speed gives

$$U_H = \frac{IB}{nec}.$$

Numerically,

$$U_H = \frac{(1.0)(0.10)}{(2.5 \times 10^{22})(1.6 \times 10^{-19})(1.0 \times 10^{-3})} = \boxed{0.025 \text{ V} = 25 \text{ mV}}.$$

The $-y$ face accumulates electrons and is at lower potential; the $+y$ face is at higher potential.

c) DC component for alternating current and magnetic field

Assume quasistatic operation: the electron drift and Hall charge distribution follow the applied signals sufficiently rapidly, and inductive pickup in the voltage leads is negligible. The instantaneous Hall voltage is then

$$U_H(t) = \frac{I(t)B(t)}{nec} = \frac{I_0 B_0}{nec} \sin \omega t \sin(\omega t + \delta).$$

Using

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)],$$

we obtain

$$U_H(t) = \frac{I_0 B_0}{2nec} [\cos \delta - \cos(2\omega t + \delta)].$$

The second term averages to zero over a cycle. Hence

$$U_{H,DC} = \overline{U_H} = \frac{I_0 B_0}{2nec} \cos \delta.$$

Equivalently,

$$\overline{U_H} = \frac{I_{\text{rms}} B_{\text{rms}}}{nec} \cos \delta.$$

The output contains a DC component and a ripple at twice the applied frequency. Its DC component vanishes when the two inputs are in quadrature.

d) Circuit for measuring AC power

The result in part c) makes the Hall element a multiplier: its transverse voltage is proportional to the product of its longitudinal current and the magnetic field.

A suitable circuit has these connections:

1. Connect a field coil in series with the apparatus. Place the Hall bar inside the coil, with its c -direction parallel to the field. Use a linear, preferably air-cored coil, so that

$$B(t) = k_I i_L(t),$$

where $i_L(t)$ is the apparatus current and k_I is the coil's field constant.

2. Connect a separate branch across the AC network, consisting of a large noninductive resistor R in series with the Hall bar's two longitudinal end contacts. The bar's longitudinal resistance follows from part a):

$$r_H = \frac{E_x a}{I_H} = \frac{a}{ne\mu bc}.$$

Thus, with negligible branch reactance,

$$I_H(t) = \frac{u(t)}{R + r_H} = k_U u(t).$$

Choosing $R \gg r_H$ gives $k_U \simeq 1/R$ and limits the sensing current.

3. Connect opposite Hall contacts on the b -faces, at the same longitudinal position, to a high-input-impedance averaging voltmeter. A buffered low-pass filter followed by a DC meter is also suitable. Average over many AC cycles.

Neglect the field coil's voltage drop and the loading of the voltage-sensing branch. Then $u(t)$ is also the apparatus voltage, and

$$U_H(t) = \frac{I_H(t)B(t)}{nec} = \frac{k_U k_I}{nec} u(t) i_L(t).$$

Its average is therefore

$$\overline{U_H} = \frac{k_U k_I}{nec} \overline{u(t) i_L(t)} = \frac{k_U k_I}{nec} P.$$

For sinusoidal operation, write

$$u(t) = U_0 \sin \omega t, \quad i_L(t) = I_{L0} \sin(\omega t - \varphi).$$

Then

$$P = \frac{U_0 I_{L0}}{2} \cos \varphi = U_{\text{rms}} I_{L,\text{rms}} \cos \varphi.$$

The meter consequently measures the average power consumed, including the power-factor dependence. Its watt calibration is

$$P = \frac{nec}{k_U k_I} \overline{U_H}.$$

The solution was checked against the official solution.

Problem 3: Space probe escaping the Solar system with the help of Mars

In a space research project two schemes of launching a space probe out of the Solar system are discussed. The first scheme (i) is to launch the probe with a velocity large enough to escape from the Solar system directly. According to the second one (ii), the probe is to approach one of the outer planets, and with its help change its direction of motion and reach the velocity necessary to escape from the Solar system. Assume that the probe moves under the gravitational field of only the Sun or the planet, depending on whichever field is stronger at that point.

- a) Determine the minimum velocity and its direction relative to the Earth's motion that should be given to the probe on launching according to scheme (i).

b) Suppose that the probe has been launched in the direction determined in a) but with another velocity. Determine the velocity of the probe when it crosses the orbit of Mars, i. e., its parallel and perpendicular components with respect to this orbit. Mars is not near the point of crossing, when crossing occurs.

c) Let the probe enter the gravitational field of Mars. Find the minimum launching velocity from the Earth necessary for the probe to escape from the Solar system.

Hint: From the result a) you know the optimal magnitude and the direction of the velocity of the probe that is necessary to escape from the Solar system after leaving the gravitational field of Mars. (You do not have to worry about the precise position of Mars during the encounter.) Find the relation between this velocity and the velocity components before the probe enters the gravitational field of Mars; i. e., the components you determined in b). What about the conservation of energy of the probe?

d) Estimate the maximum possible fractional saving of energy in scheme (ii) with respect to scheme (i). Notes: Assume that all the planets revolve round the Sun in circles, in the same direction and in the same plane. Neglect the air resistance, the rotation of the Earth around its axis as well as the energy used in escaping from the Earth's gravitational field.

Data: Velocity of the Earth round the Sun is 30 km/s, and the ratio of the distances of the Earth and Mars from the Sun is 2/3.

Official English text of the problem.

Answers

Let $V = 30 \text{ km s}^{-1}$, and let u denote the launch speed relative to Earth.

- **a)** Launch in the direction of Earth's orbital motion, with

$$u_{i,\min} = (\sqrt{2} - 1)V = 12.4 \text{ km s}^{-1}.$$

- **b)** For a prograde launch with speed u , the heliocentric components at Mars's orbit are

$$v_{\parallel} = \frac{2}{3}(V + u), \quad v_{\perp} = \sqrt{\frac{5}{9}(V + u)^2 - \frac{2}{3}V^2},$$

where $v_{\perp}$ points radially outward on the first crossing. Reaching this orbit requires $u \geq 2.86 \text{ km s}^{-1}$.

- **c)** With $V_M = \sqrt{2/3} V$, the minimum Earth-relative launch speed is

$$u_{ii,\min} = V_M \left[\frac{2}{3} + \sqrt{\frac{4}{9} + (\sqrt{2} - 1)^2} \right] - V = 5.56 \text{ km s}^{-1}.$$

The launch is prograde; the optimal outgoing velocity is also prograde and tangential to Mars's orbit.

- **d)** The maximum fractional launch-energy saving is

$$\frac{E_i - E_{ii}}{E_i} = 1 - \left(\frac{u_{ii,\min}}{u_{i,\min}} \right)^2 \simeq 0.800,$$

or approximately 80%.

Solution

Write the Earth's and Mars's orbital radii as r and R , respectively, and put $\mu = GM_\odot$. The supplied ratio and the circular-orbit condition give

$$\frac{r}{R} = \frac{2}{3}, \quad V^2 = \frac{\mu}{r}.$$

All probe velocities below are heliocentric unless explicitly stated otherwise. Earth's gravitational field is neglected as instructed.

a)

The probe escapes from the Sun if its specific mechanical energy is nonnegative:

$$\varepsilon = \frac{1}{2}v_0^2 - \frac{\mu}{r} \geq 0.$$

Thus its initial heliocentric speed must satisfy

$$v_0 \geq \sqrt{\frac{2\mu}{r}} = \sqrt{2}V.$$

For a given Earth-relative launch speed u , the largest possible heliocentric speed is $V + u$, obtained by launching parallel to Earth's motion. Therefore

$$V + u_{i,\min} = \sqrt{2}V,$$

so

$$u_{i,\min} = (\sqrt{2} - 1)V = 12.4 \text{ km s}^{-1}.$$

The corresponding heliocentric speed is 42.4 km s^{-1} . At this threshold the trajectory is parabolic, with zero speed at infinity.

b)

For a prograde launch, put

$$v_0 = V + u.$$

The initial velocity is tangential to Earth's orbit.

Between Earth's and Mars's orbits, the Sun's force is central, so angular momentum about the Sun is conserved:

$$rv_0 = Rv_{\parallel}.$$

Consequently,

$$v_{\parallel} = \frac{2}{3}v_0 = \frac{2}{3}(V + u).$$

Conservation of mechanical energy gives

$$\frac{1}{2}v_0^2 - \frac{\mu}{r} = \frac{1}{2}(v_{\parallel}^2 + v_{\perp}^2) - \frac{\mu}{R}.$$

Using $\mu/r = V^2$ and $\mu/R = \frac{2}{3}V^2$,

$$v_{\parallel}^2 + v_{\perp}^2 = v_0^2 - \frac{2}{3}V^2.$$

Subtracting $v_{\parallel}^2 = \frac{4}{9}v_0^2$,

$$v_{\perp} = \sqrt{\frac{5}{9}v_0^2 - \frac{2}{3}V^2} = \sqrt{\frac{5}{9}(V+u)^2 - \frac{2}{3}V^2}.$$

This is the outward radial component at the first crossing. Its sign reverses at an inward crossing.

For the probe to reach Mars's orbital radius, the radicand must be nonnegative:

$$v_0^2 \geq \frac{6}{5}V^2.$$

Hence

$$u \geq \left(\sqrt{\frac{6}{5}} - 1\right)V = 2.86 \text{ km s}^{-1}.$$

Equality corresponds to an orbit whose aphelion just touches Mars's orbit.

c)

Mars's orbital speed is

$$V_M = \sqrt{\frac{\mu}{R}} = \sqrt{\frac{2}{3}}V = 24.5 \text{ km s}^{-1}.$$

Treat the encounter as localized and short compared with Mars's orbital motion, so that Mars's velocity is effectively constant. In the Mars frame, the incoming probe velocity has components

$$w_{\parallel} = v_{\parallel} - V_M, \quad w_{\perp} = v_{\perp}.$$

Its magnitude is therefore

$$w^2 = (v_{\parallel} - V_M)^2 + v_{\perp}^2.$$

In this frame the gravitational encounter conserves mechanical energy. At equal distances from Mars before and after the encounter, the relative speed is the same, although its direction changes.

If the outgoing relative velocity makes an angle θ with Mars's motion, the outgoing heliocentric speed satisfies

$$v_{\text{out}}^2 = V_M^2 + w^2 + 2V_M w \cos \theta.$$

It is largest when $\theta = 0$, giving

$$v_{\text{out,max}} = V_M + w.$$

Thus the optimal encounter directs the outgoing relative velocity along Mars's motion.

The solar escape speed at Mars's orbital radius is

$$v_{\text{esc}}(R) = \sqrt{\frac{2\mu}{R}} = \sqrt{2} V_M.$$

The minimum required relative speed is consequently

$$w_{\min} = (\sqrt{2} - 1)V_M.$$

Now use the components found in part b):

$$\begin{aligned} w^2 &= v_{\parallel}^2 + v_{\perp}^2 - 2V_M v_{\parallel} + V_M^2 \\ &= v_0^2 - \frac{2}{3}V^2 - \frac{4}{3}V_M v_0 + V_M^2. \end{aligned}$$

Since $V_M^2 = \frac{2}{3}V^2$, this simplifies to

$$w^2 = v_0^2 - \frac{4}{3}V_M v_0.$$

At the escape threshold,

$$v_0^2 - \frac{4}{3}V_M v_0 = (\sqrt{2} - 1)^2 V_M^2.$$

The positive root is

$$v_{0,\min} = V_M \left[\frac{2}{3} + \sqrt{\frac{4}{9} + (\sqrt{2} - 1)^2} \right] = 35.555 \dots \text{ km s}^{-1}.$$

Therefore the required launch speed relative to Earth is

$$u_{\text{ii},\min} = v_{0,\min} - V = 5.56 \text{ km s}^{-1}.$$

This exceeds the 2.86 km s^{-1} needed to reach Mars's orbit, so the incoming trajectory is possible.

The prograde launch also minimizes the required speed over possible launch directions. To see this, let α be the launch angle relative to Earth's motion and write $q = r/R = 2/3$. The same energy and angular-momentum relations give

$$w^2 = u^2 + 2Vu(1 - q^{3/2}) \cos \alpha + V^2(3q - 1 - 2q^{3/2}).$$

For fixed u , this is greatest at $\cos \alpha = 1$, since $q < 1$. Thus launching prograde provides the largest available encounter-relative speed for a given launch speed.

Numerically, the optimal incoming trajectory has

$$v_{\parallel} = 23.70 \text{ km s}^{-1}, \quad v_{\perp} = 10.12 \text{ km s}^{-1}.$$

These give $w = 10.15 \text{ km s}^{-1}$. After the optimal turn,

$$v_{\text{out}} = V_M + w = 34.64 \text{ km s}^{-1} = \sqrt{2} V_M,$$

as required.

There is no violation of energy conservation. The probe's heliocentric energy increases during the encounter by

$$\Delta E = m \mathbf{V}_M \cdot (\mathbf{w}_{\text{out}} - \mathbf{w}_{\text{in}}).$$

For the optimal turn this becomes

$$\Delta E = m V_M [w - (v_{\parallel} - V_M)] > 0.$$

This energy comes from Mars's orbital motion. Mars's corresponding change in velocity is negligible because its mass is much greater than the probe's.

d)

With terrestrial escape energy neglected, the energy imparted at launch in Earth's frame is

$$E_{\text{launch}} = \frac{1}{2} m u^2.$$

Hence the maximum fractional saving is

$$\begin{aligned} f_{\text{max}} &= \frac{E_i - E_{\text{ii}}}{E_i} \\ &= 1 - \left(\frac{u_{\text{ii,min}}}{u_{\text{i,min}}} \right)^2 \\ &= 1 - \left(\frac{5.55519}{12.42641} \right)^2 \\ &= 0.80015. \end{aligned}$$

Thus

$$\boxed{f_{\text{max}} \simeq 80\%.$$

The corresponding launch energies per unit probe mass are approximately 77.2 MJ kg^{-1} for direct escape and 15.4 MJ kg^{-1} with the ideal Mars assist.

The solution was checked against the official solution.