

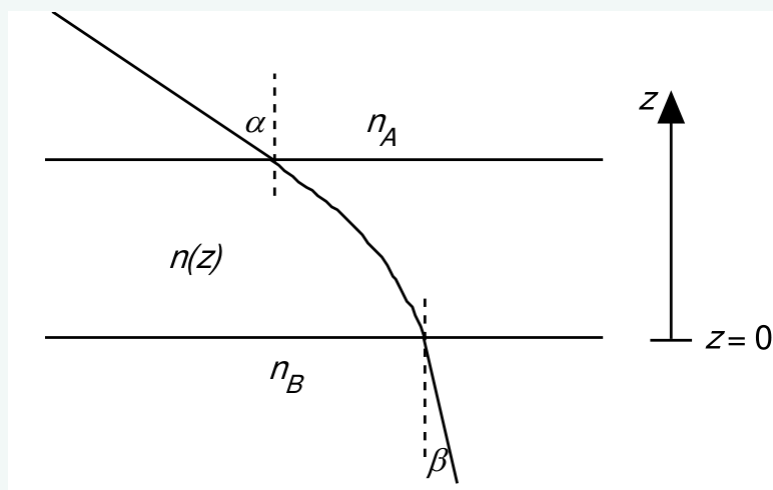
IPhO 1984

Theory problems and solutions · Sigtuna, Sweden · 3 theory problems with complete solutions

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Problem 1: Refraction in a layered medium and the desert mirage

a) Consider a plane-parallel transparent plate, where the refractive index, n , varies with distance, z , from the lower surface (see figure). Show that $n_A \sin \alpha = n_B \sin \beta$. The notation is that of the figure.



b) Assume that you are standing in a large flat desert. At some distance you see what appears to be a water surface. When you approach the "water" it seems to move away such that the distance to the "water" is always constant. Explain the phenomenon.

c) Compute the temperature of the air close to the ground in b) assuming that your eyes are located 1.60 m above the ground and that the distance to the "water" is 250 m. The refractive index of the air at 15°C and at normal air pressure (101.3 kPa) is 1.000276. The temperature of the air more than 1 m above the ground is assumed to be constant and equal to 30°C . The atmospheric pressure is assumed to be normal. The refractive index, n , is such that $n - 1$ is proportional to the density of the air. Discuss the accuracy of your result.

Official English text of the problem.

Answers

- **a)** The conserved quantity is $n(z) \sin \theta(z)$, where θ is the angle to the normal. Consequently,

$$n_A \sin \alpha = n_B \sin \beta.$$

- **b)** This is an **inferior mirage**: hot, less dense air near the ground bends sky light upward into the observer's eyes. The apparent water remains at a constant distance because the horizontal temperature distribution and the observer's eye height remain unchanged.
- **c)** Interpreting 250 m as the apparent distance to the mirage's near edge,

$$T_{\text{ground air}} \simeq 329 \text{ K} \simeq 56^\circ \text{C}.$$

The small-angle approximation contributes less than 0.001°C ; uncertainty in the apparent edge and departures from ideal atmospheric layering are more significant.

Solution

a)

Divide the plate into many thin horizontal layers, each having an approximately constant refractive index. Let n_i be the index of layer i , and let θ_i be the ray's angle to the normal there.

Snell's law at each interface gives

$$n_i \sin \theta_i = n_{i+1} \sin \theta_{i+1}.$$

Since all interfaces are parallel, their normals have the same direction. Including the entrance and exit interfaces,

$$n_A \sin \alpha = n_1 \sin \theta_1 = \cdots = n_N \sin \theta_N = n_B \sin \beta.$$

Taking the limit of infinitesimally thin layers establishes the result for a continuously varying index:

$$n(z) \sin \theta(z) = p = \text{constant}, \quad n_A \sin \alpha = n_B \sin \beta.$$

b)

Sunlight heats the desert surface, which heats the air immediately above it. At approximately constant pressure, this warmer air has a lower density and hence a smaller refractive index. Thus the refractive index increases with height through the hot layer.

Consider a ray of sky light travelling downward at a shallow angle to the ground. From part a),

$$\sin \theta(z) = \frac{p}{n(z)}.$$

As the ray enters air of smaller refractive index, its angle θ to the vertical increases. If it reaches a height where $n(z) = p$, it becomes horizontal. It then turns upward toward regions of larger refractive index and can reach the observer.

The observer traces the arriving light backward along a straight line and therefore sees sky light apparently coming from below the horizon. This resembles the reflection of the sky in water. In a smooth temperature gradient, the turning occurs through continuous refraction.

If the atmosphere is horizontally uniform, moving horizontally leaves the optical situation unchanged. At a fixed eye height, the angular position of the mirage's near edge—and therefore its apparent distance—remains constant.

c)

Let n_e denote the refractive index at eye height and n_g the minimum index in the hottest air next to the ground. Assume geometrical optics, horizontal stratification, and a temperature that increases toward the ground.

First determine the refractive index as a function of temperature. For air treated as an ideal gas at constant pressure,

$$\rho \propto \frac{1}{T},$$

where T is the absolute temperature. Since $n - 1 \propto \rho$,

$$n(T) - 1 = \frac{A}{T},$$

with

$$A = (1.000276 - 1)(273.15 + 15) = 0.0795294 \text{ K}.$$

The eyes are above the 1 m level, so their surrounding air is at

$$T_e = 303.15 \text{ K}.$$

Consequently,

$$n_e = 1 + \frac{0.0795294}{303.15} = 1.000262343.$$

Let δ be the depression angle of the mirage's near edge below the horizontal. Interpreting the stated distance as the apparent ground-plane distance, straight backward projection of the viewing direction gives

$$\tan \delta = \frac{h}{D} = \frac{1.60}{250} = 0.00640,$$

so

$$\delta = 0.0063999 \text{ rad} \simeq 0.367^\circ.$$

The near edge is determined by the limiting ray that becomes horizontal in the minimum-index air. At eye height its angle to the vertical is $90^\circ - \delta$; at its turning point that angle is 90° . Part a) therefore gives

$$n_e \sin(90^\circ - \delta) = n_g \sin 90^\circ,$$

or

$$n_g = n_e \cos \delta = \frac{n_e}{\sqrt{1 + (h/D)^2}}.$$

Rays with a smaller depression angle turn above the ground and bring sky light to the observer. Rays with a larger depression angle reach the ground.

Numerically,

$$n_g = \frac{1.000262343}{\sqrt{1 + (1.60/250)^2}} = 1.000241859.$$

Thus

$$T_g = \frac{A}{n_g - 1} = \frac{0.0795294}{0.000241859} = 328.826 \text{ K},$$

and hence

$$T_g \simeq 56^\circ\text{C}.$$

For the accuracy, the relevant refractive-index decrease is

$$n_e - n_g \simeq 2.05 \times 10^{-5}.$$

The angle is sufficiently small that replacing

$$\cos \delta = \frac{1}{\sqrt{1 + (h/D)^2}}$$

by $1 - \frac{1}{2}(h/D)^2$ changes the calculated temperature by only about 0.0009 K.

The observational and atmospheric assumptions matter more. Differentiating the temperature formula, while holding the reference index and upper-air temperature fixed, gives at the stated values

$$dT_g \simeq 55.7 \text{ K} \left(\frac{dh}{h} - \frac{dD}{D} \right).$$

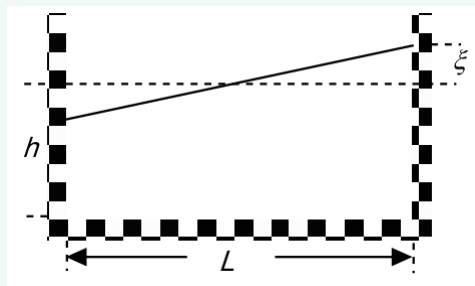
Thus a 1% error in h/D changes the inferred temperature by approximately 0.56°C .

A blurred mirage edge, turbulence, uneven heating, and departures from horizontal layering can introduce larger errors. The detailed vertical profile is unnecessary for the ideal limiting-angle calculation, but a distinct edge associated with the hottest air is required. Since measurement uncertainties are not supplied, a numerical error bar is not justified.

The solution was checked against the official solution.

Problem 2: Seiching of lake water

In certain lakes there is a strange phenomenon called "seiching" which is an oscillation of the water. Lakes in which you can see this phenomenon are normally long compared with the depth and also narrow. It is natural to see waves in a lake but not something like the seiching, where the entire water volume oscillates, like the coffee in a cup that you carry to a waiting guest.



In order to create a model of the seiching we look at water in a rectangular container. The length of the container is L and the depth of the water is h . Assume that the surface of the water to begin with makes a small angle with the horizontal. The seiching will then start, and we assume that the water surface continues to be plane but oscillates around an axis in the horizontal plane and located in the middle of the container.

Create a model of the movement of the water and derive a formula for the oscillation period T . The starting conditions are given in figure above. Assume that $\xi \ll h$. The table below shows experimental oscillation periods for different water depths in two containers of different lengths. Check in some reasonable way how well the formula that you have derived agrees with the experimental data. Give your opinion on the quality of your model.

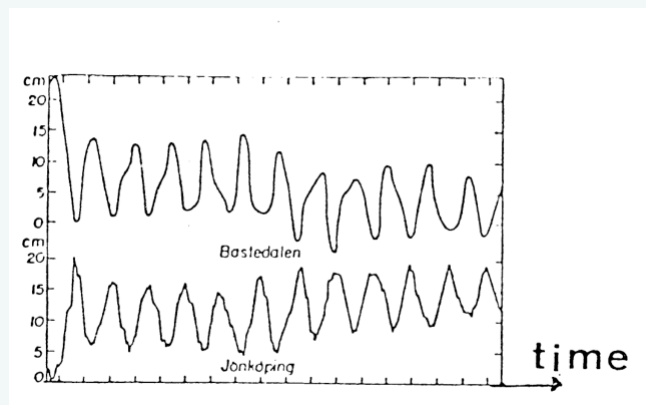
Table 1. $L = 479 \text{ mm}$

h/mm	30	50	69	88	107	124	142
T/s	1.78	1.40	1.18	1.08	1.00	0.91	0.82

Table 2. $L = 143 \text{ mm}$

h/mm	31	38	58	67	124
T/s	0.52	0.52	0.43	0.35	0.28

The graph below shows results from measurements in lake Vättern in Sweden. This lake has a length of 123 km and a mean depth of 50 m. What is the time scale in the graph?



The water surface level in Bastudalen (northern end of lake Vättern) and Jönköping (southern end).

Official English text of the problem.

Answers

Problem 2 has no numbered subparts or printed point values. Its requests are answered in their printed order.

- **2 — Motion and period:** A simple model using the centre of mass and hydrostatic end-wall forces gives harmonic oscillations with

$$T_0 = \frac{\pi L}{\sqrt{3gh}}.$$

An energy model accounting for the variation of horizontal flow speed gives the refined estimate

$$T_1 = \frac{2\pi L}{\sqrt{10gh}} \simeq 1.987 \frac{L}{\sqrt{gh}}.$$

- **2 — Experimental comparison and quality:** T_0 underestimates the measured periods by 10.0–15.2% in the longer container and 8.6–20.0% in the shorter one. The model captures the approximate scaling $T \propto L/\sqrt{h}$. The energy estimate agrees more closely: its largest discrepancies are 7.1% and 12.4%, respectively. Both models have limited validity when the depth becomes comparable with the length.
- **2 — Lake Vättern:** $T_0 \simeq 2.8$ h, while $T_1 \simeq 3.1$ h. Thus the graph shows approximately **three hours per oscillation**, or **two hours per horizontal tick interval**, with about **a day and a half** represented overall.

Solution

2 — Model of the motion and oscillation period

Let the container have width b , and let the water density be ρ . Measure x horizontally from the middle of the container. Write the surface height above the bottom as

$$H(x, t) = h + \eta(x, t), \quad \eta(x, t) = \frac{2q(t)x}{L}, \quad -\frac{L}{2} \leq x \leq \frac{L}{2}.$$

Thus the depths at the left and right ends are $h - q$ and $h + q$, respectively. Initially $q = \xi$.

We neglect viscosity, damping and surface tension. For a long, shallow container, vertical accelerations can initially be neglected and the pressure approximated as hydrostatic. Small amplitude, $|q| \ll h$, allows a linear oscillation model.

The total mass is

$$M = \rho b L h,$$

since the volume above the mean surface equals the missing volume below it. The horizontal coordinate X of the centre of mass is

$$\begin{aligned} X &= \frac{\rho b}{M} \int_{-L/2}^{L/2} x H(x, t) dx \\ &= \frac{1}{Lh} \frac{2q}{L} \int_{-L/2}^{L/2} x^2 dx \\ &= \boxed{\frac{Lq}{6h}}. \end{aligned}$$

The hydrostatic force exerted by an end wall on water of depth H has magnitude

$$b \int_0^H \rho g (H - z) dz = \frac{1}{2} \rho g b H^2.$$

Consequently, the net horizontal force on the water is

$$\begin{aligned} F_x &= \frac{1}{2} \rho g b [(h - q)^2 - (h + q)^2] \\ &= -2 \rho g b h q. \end{aligned}$$

The force is restoring: when the water is higher on the right, it acts towards the left. Newton's second law gives

$$M \ddot{X} = F_x,$$

or

$$\rho b L h \frac{L}{6h} \ddot{q} = -2 \rho g b h q.$$

Therefore,

$$\ddot{q} + \frac{12gh}{L^2} q = 0.$$

This is simple harmonic motion, with

$$\omega_0 = \frac{\sqrt{12gh}}{L}, \quad T_0 = \frac{2\pi}{\omega_0} = \frac{\pi L}{\sqrt{3gh}}.$$

If the illustrated configuration is released from rest,

$$q(t) = \xi \cos(\omega_0 t), \quad \eta(x, t) = \frac{2\xi x}{L} \cos(\omega_0 t).$$

The two ends therefore oscillate in opposite phase. The period is independent of the small amplitude, the width and the density.

An energy estimate accounting for the flow speed

The planar surface is an approximation to the actual motion. We can assess the effect of resolving the internal flow by constructing an energy model.

Assume that the horizontal velocity $u(x, t)$ is approximately uniform over the depth. Conservation of volume, linearized about the depth h , requires

$$\frac{\partial \eta}{\partial t} + h \frac{\partial u}{\partial x} = 0.$$

Using $\eta = 2qx/L$ and imposing zero horizontal flow through both end walls gives

$$u(x, t) = \frac{\dot{q}}{Lh} \left(\frac{L^2}{4} - x^2 \right).$$

The horizontal speed is greatest at the middle and vanishes at the walls. In the shallow-water approximation, the kinetic energy of vertical motion is small compared with that of horizontal motion. Thus

$$K = \frac{1}{2} \rho b h \int_{-L/2}^{L/2} u^2 dx$$

$$= \boxed{\frac{\rho b L^3}{60 h} \dot{q}^2}.$$

The increase in gravitational potential energy above that of a horizontal surface is

$$U = \frac{\rho g b}{2} \int_{-L/2}^{L/2} [(h + \eta)^2 - h^2] dx$$

$$= \frac{\rho g b}{2} \int_{-L/2}^{L/2} \eta^2 dx$$

$$= \boxed{\frac{\rho g b L}{6} q^2},$$

where the term proportional to $\int \eta dx$ vanishes by volume conservation.

Conserving $K + U$ now gives

$$\ddot{q} + \frac{10gh}{L^2} q = 0,$$

and hence

$$\boxed{T_1 = \frac{2\pi L}{\sqrt{10gh}} = \sqrt{\frac{6}{5}} T_0.}$$

This estimate is about 9.5% longer than the simple centre-of-mass result.

The differing coefficients reflect the approximate descriptions of the motion: the first model uses hydrostatic end forces, while the second uses the energy of a volume-conserving trial flow. The actual free surface generally departs from a plane, so neither coefficient should be regarded as an exact hydrodynamic result.

2 — Comparison with the measurements and quality of the model

Using $g = 9.81 \text{ m s}^{-2}$, the predicted periods are:

For $L = 479 \text{ mm}$:

$h \text{ (mm)}$	Measured $T \text{ (s)}$	$T_0 \text{ (s)}$	$T_1 \text{ (s)}$
30	1.78	1.60	1.75
50	1.40	1.24	1.36
69	1.18	1.06	1.16
88	1.08	0.935	1.02
107	1.00	0.848	0.929
124	0.91	0.788	0.863
142	0.82	0.736	0.806

For $L = 143 \text{ mm}$:

h (mm)	Measured T (s)	T_0 (s)	T_1 (s)
31	0.52	0.470	0.515
38	0.52	0.425	0.465
58	0.43	0.344	0.377
67	0.35	0.320	0.350
124	0.28	0.235	0.258

For a quantitative comparison, define the relative discrepancy as

$$\varepsilon = \frac{T_{\text{model}} - T_{\text{measured}}}{T_{\text{measured}}}.$$

The simple model gives periods systematically below the observations: by 10.0–15.2% in the longer container and 8.6–20.0% in the shorter one. The energy estimate reduces the maximum absolute discrepancies to 7.1% and 12.4%.

The approximate scaling can also be checked through the dimensionless quantity

$$C = \frac{T\sqrt{gh}}{L}.$$

The experimental values range from approximately 2.02 to 2.14 in the longer container and 1.98 to 2.27 in the shorter one. These are reasonably close to a common constant. The two model predictions are

$$C_0 = \frac{\pi}{\sqrt{3}} \simeq 1.814, \quad C_1 = \frac{2\pi}{\sqrt{10}} \simeq 1.987.$$

Thus the models provide a useful description of the scale and parameter dependence, and resolving the flow speed improves agreement with these data.

Their validity nevertheless has limits. The ratios h/L range from 0.063 to 0.30 in the longer container and from 0.22 to 0.87 in the shorter one. In particular, the deepest measurements in the shorter container fall outside a reliable shallow-water approximation. Vertical motion and departures from the assumed planar surface then matter. No experimental uncertainties are supplied, so the table establishes numerical agreement rather than a statistical validation.

2 — Time scale of the lake measurements

For lake Vättern,

$$L = 1.23 \times 10^5 \text{ m}, \quad h = 50 \text{ m},$$

giving

$$\sqrt{gh} \simeq 22.1 \text{ m s}^{-1}.$$

The simple model predicts

$$T_0 = \frac{\pi(1.23 \times 10^5)}{\sqrt{3(9.81)(50)}} \simeq 1.01 \times 10^4 \text{ s} = \boxed{2.8 \text{ h}}.$$

The energy estimate gives

$$T_1 = \frac{2\pi(1.23 \times 10^5)}{\sqrt{10(9.81)(50)}} \simeq 1.10 \times 10^4 \text{ s} = \boxed{3.1 \text{ h}}.$$

Accordingly, successive peaks in either trace are separated by approximately **three hours**.

In the reproduced graph, a complete oscillation occupies approximately 1.5 horizontal tick intervals. Therefore,

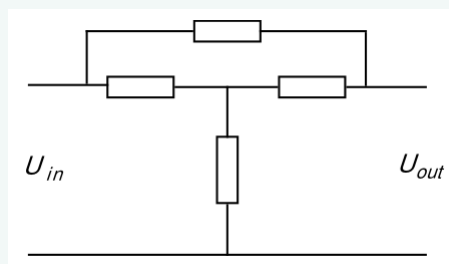
$$\text{time per tick interval} \approx \frac{3 \text{ h}}{1.5} \approx \boxed{2 \text{ h}}.$$

There are roughly twelve oscillations across the record, corresponding to approximately 36 h, or a day and a half. The two traces are approximately in opposite phase, as expected for oscillations at opposite ends of the lake.

The solution was checked against the official solution.

Problem 3: Electronic frequency filter

An electronic frequency filter consists of four components coupled as in the upper figure. The impedance of the source can be neglected and the impedance of the load can be taken as infinite. The filter should be such that the voltage ratio U_{out}/U_{in} has a frequency dependence shown in the lower where U_{in} is the input voltage and U_{out} is the output voltage. At frequency f_0 the phase lag between the two voltages is zero.

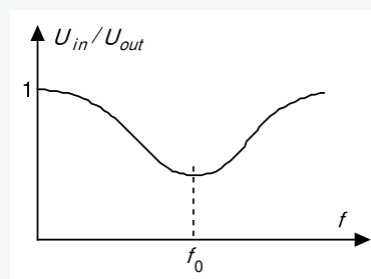


In order to build the filter you can choose from the following components:

2 resistors, 10 k Ω

2 capacitors, 10 nF

2 solenoids, 160 mH (iron-free and with negligible resistance)



Construct, by combining four of these components, a filter that fulfils the stated conditions. Determine the frequency f_0 and the ratio U_{out}/U_{in} at this frequency for as many component combinations as possible.

Official English text of the problem.

Answers

- **Problem 3.** There are **12 valid component placements**, interpreting the plotted dip as the magnitude of $U_{\text{out}}/U_{\text{in}}$, as specified in the prose.

Define the positions, in order, as

$$(Z_1, Z_2, Z_3, Z_4) = (\text{left horizontal, right horizontal, upper bridge, vertical shunt}).$$

Each R denotes a $10\text{ k}\Omega$ resistor, each C a 10 nF capacitor, and each L a 160 mH solenoid.

Component placement (Z_1, Z_2, Z_3, Z_4)	f_0	$U_{\text{out}}/U_{\text{in}}$ at f_0
(R, R, C, C) or (C, C, R, R)	1.59 kHz	$2/3 \approx 0.667$
(R, R, L, L) or (L, L, R, R)	9.95 kHz	$2/3 \approx 0.667$
(R, L, R, C)	5.63 kHz	$4/29 \approx 0.138$
(R, C, R, L)	2.81 kHz	$4/29 \approx 0.138$
(C, R, L, R)	5.63 kHz	$25/29 \approx 0.862$
(L, R, C, R)	2.81 kHz	$25/29 \approx 0.862$
(L, C, R, L) or (L, C, C, R)	5.63 kHz	$1/2$
(C, L, R, C) or (C, L, L, R)	2.81 kHz	$1/2$

For every placement, the output and input are **in phase at f_0** .

Solution

The problem has no numbered subparts. We first derive the response of the given circuit, then identify the component placements and calculate their minima.

Use ideal components with

$$R = 1.00 \times 10^4 \Omega, \quad C = 1.00 \times 10^{-8} \text{ F}, \quad L = 0.160 \text{ H}.$$

In sinusoidal steady state, using phasors with time dependence $e^{i\omega t}$,

$$Z_R = R, \quad Z_L = i\omega L, \quad Z_C = \frac{1}{i\omega C}, \quad \omega = 2\pi f.$$

The negligible source impedance fixes the input voltage. The infinite load impedance means that no external current leaves the output terminal.

Let V_m be the voltage at the junction of Z_1, Z_2, Z_4 , measured relative to the common lower conductor. Kirchhoff's current law gives

$$\begin{aligned} \frac{V_m - V_{\text{in}}}{Z_1} + \frac{V_m - V_{\text{out}}}{Z_2} + \frac{V_m}{Z_4} &= 0, \\ \frac{V_{\text{out}} - V_m}{Z_2} + \frac{V_{\text{out}} - V_{\text{in}}}{Z_3} &= 0. \end{aligned}$$

Eliminating V_m , the complex voltage ratio is

$$H(\omega) \equiv \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_1 Z_2 + Z_4 (Z_1 + Z_2 + Z_3)}{Z_1 Z_2 + Z_4 (Z_1 + Z_2 + Z_3) + Z_1 Z_3}.$$

It is convenient to write this as

$$H = \frac{K}{K+1},$$

where

$$K = \frac{Z_2}{Z_3} + \frac{Z_4}{Z_3} + \frac{Z_2 Z_4}{Z_1 Z_3} + \frac{Z_4}{Z_1}.$$

Component placements. Define the dimensionless constant

$$\kappa = \frac{L}{R^2 C}.$$

For each of the following placements, substitution into K gives

$$K = \alpha + i \left(\beta \omega - \frac{\gamma}{\omega} \right), \quad \alpha, \beta, \gamma > 0.$$

The component codes in the table are written in the order $Z_1 Z_2 Z_3 Z_4$.

Placement	α	β	γ
$RRCC, CCRR$	2	RC	$1/(RC)$
$RRLL, LLRR$	2	L/R	R/L
$RLRC$	κ	L/R	$2/(RC)$
$RCRL$	κ	$2L/R$	$1/(RC)$
$CRLR$	$1/\kappa$	RC	$2R/L$
$LRCR$	$1/\kappa$	$2RC$	R/L
$LCRL$	1	L/R	$2/(RC)$
$LCCR$	1	RC	$2R/L$
$CLRC$	1	$2L/R$	$1/(RC)$
$CLLR$	1	$2RC$	R/L

For example, the placement $RLRC$ means

$$Z_1 = R, \quad Z_2 = i\omega L, \quad Z_3 = R, \quad Z_4 = \frac{1}{i\omega C}.$$

It gives

$$K = \frac{i\omega L}{R} + \frac{1}{i\omega CR} + \frac{L}{CR^2} + \frac{1}{i\omega CR} = \kappa + i \left(\frac{\omega L}{R} - \frac{2}{\omega CR} \right),$$

as listed.

Frequency dependence and phase. Put

$$u = \beta \omega - \frac{\gamma}{\omega}.$$

Then

$$H = \frac{\alpha + iu}{\alpha + 1 + iu}, \quad |H|^2 = \frac{\alpha^2 + u^2}{(\alpha + 1)^2 + u^2}.$$

Since

$$\frac{d|H|^2}{d(u^2)} = \frac{2\alpha + 1}{[(\alpha + 1)^2 + u^2]^2} > 0,$$

the magnitude is smallest when $u = 0$. Moreover,

$$\frac{du}{d\omega} = \beta + \frac{\gamma}{\omega^2} > 0,$$

so there is exactly one minimum at positive frequency:

$$\boxed{\omega_0 = \sqrt{\frac{\gamma}{\beta}}}, \quad \boxed{f_0 = \frac{1}{2\pi} \sqrt{\frac{\gamma}{\beta}}}.$$

At this frequency,

$$\boxed{H(\omega_0) = \frac{\alpha}{\alpha + 1} > 0}.$$

Thus the voltage ratio is real and positive: the phase difference is zero, as required.

At both frequency limits,

$$\omega \rightarrow 0 : \quad u \rightarrow -\infty, \quad \omega \rightarrow \infty : \quad u \rightarrow +\infty,$$

and consequently

$$\lim_{\omega \rightarrow 0} H = \lim_{\omega \rightarrow \infty} H = 1.$$

Every listed placement therefore has the required curve: its magnitude starts at unity, decreases smoothly to a single positive minimum at f_0 , and then rises towards unity.

Numerical frequencies and voltage ratios. The table produces four distinct frequencies:

$$\begin{aligned} f_{RC} &= \frac{1}{2\pi RC} = 1.5915 \times 10^3 \text{ Hz} \approx \boxed{1.59 \text{ kHz}}, \\ f_{RL} &= \frac{R}{2\pi L} = 9.9472 \times 10^3 \text{ Hz} \approx \boxed{9.95 \text{ kHz}}, \\ f_+ &= \frac{1}{2\pi} \sqrt{\frac{2}{LC}} = 5.6270 \times 10^3 \text{ Hz} \approx \boxed{5.63 \text{ kHz}}, \\ f_- &= \frac{1}{2\pi\sqrt{2LC}} = 2.8135 \times 10^3 \text{ Hz} \approx \boxed{2.81 \text{ kHz}}. \end{aligned}$$

Also,

$$\kappa = \frac{0.160}{(10^4)^2(10^{-8})} = 0.160 = \frac{4}{25}.$$

The possible minimum ratios are therefore

$$\begin{aligned} \alpha = 2 : \quad H_0 &= \frac{2}{3}, \\ \alpha = \kappa : \quad H_0 &= \frac{\kappa}{1 + \kappa} = \frac{L}{R^2 C + L} = \frac{4}{29}, \\ \alpha = \frac{1}{\kappa} : \quad H_0 &= \frac{1}{1 + \kappa} = \frac{R^2 C}{R^2 C + L} = \frac{25}{29}, \\ \alpha = 1 : \quad H_0 &= \frac{1}{2}. \end{aligned}$$

Combining these expressions with the placement table gives all the answers above.

Completeness of the placements. There are 54 assignments of four component types to the four positions, allowing at most two of each type: 18 use two components of each of two types, and 36 use all three types.

The unity limits at both ends require K to diverge as both $\omega \rightarrow 0$ and $\omega \rightarrow \infty$. This gives a simple way to check all assignments. Write each impedance as a positive constant times s^p , with

$$s = i\omega, \quad p_R = 0, \quad p_L = 1, \quad p_C = -1.$$

The four terms of K have powers

$$p_2 - p_3, \quad p_4 - p_3, \quad p_2 + p_4 - p_1 - p_3, \quad p_4 - p_1.$$

The endpoint condition holds precisely when these powers include both a positive and a negative value. It leaves 22 placements: the 12 already obtained and the following ten.

Remaining placements	Reason they fail
$RLCC, RCLL, LRCC, CRLl$	Their only finite positive frequency with real H is $\omega = 1/\sqrt{LC}$, where $H = 0$. The output phase is then undefined, and the minimum is zero.
$LLRC, LLCR, CCRL, CCLR$	Their only finite positive frequency with real H gives $H = -1$: the phase difference is 180° .
$LLCC, CCLL$	They have undamped resonant poles and do not give the illustrated smooth dip.

For the last two placements, writing $x = \omega\sqrt{LC}$ explicitly gives

$$H = \frac{(1 - x^2)^2}{1 - 3x^2 + x^4},$$

whose denominator vanishes at

$$x^2 = \frac{3 \pm \sqrt{5}}{2}.$$

Thus the 12 listed placements exhaust the possibilities satisfying the interpreted graph and the zero-phase condition.

The solution was checked against the official solution.