

IPhO 1981

Theory problems and solutions · Varna, Bulgaria · 3 theory problems with complete solutions

eulersolve.org/ipho/1981/

Problem 1: Container emptied by a released piston and escaping gas

A static container of mass M and cylindrical shape is placed in vacuum. One of its ends is closed. A fixed piston of mass m and negligible width separates the volume of the container into two equal parts. The closed part contains n moles of monoatomic perfect gas with molar mass M_0 and temperature T . After releasing of the piston, it leaves the container without friction. After that the gas also leaves the container. What is the final velocity of the container?

The gas constant is R . The momentum of the gas up to the leaving of the piston can be neglected. There is no heat exchange between the gas, container and the piston. The change of the temperature of the gas, when it leaves the container, can be neglected. Do not account for the gravitation of the Earth.

Official English text of the problem.

Answers

- **Problem 1** — Under the free-molecular escape approximation described below, the container's final speed is

$$v_f = \sqrt{\frac{3mnRT}{M(M+m)}(1 - 2^{-2/3})} + \frac{n}{M} \sqrt{\frac{2M_0RT}{\pi 2^{2/3}}}$$

directed **toward its closed end**. The statement supplies no numerical values. The second term holds for the free-molecular escape model stated in the solution.

Solution

Take the positive x -direction along the cylinder toward its open end.

Expansion while the piston remains inside.

Let the initial gas volume be V_0 . When the piston reaches the open end, the gas occupies $2V_0$.

For a monoatomic ideal gas,

$$C_V = \frac{3}{2}R, \quad \gamma = \frac{5}{3}.$$

Using the uniform-pressure, quasistatic approximation associated with neglecting gas inertia, the expansion is adiabatic:

$$nC_V dT = -p dV = -\frac{nRT}{V} dV.$$

Consequently,

$$TV^{2/3} = \text{constant},$$

so the temperature when the piston leaves is

$$T_1 = T 2^{-2/3}.$$

Write the container velocity at that instant as $-v_1$, with $v_1 > 0$, and the piston velocity as $u_1 > 0$.

Conservation of momentum, with the stipulated neglect of gas momentum, gives

$$-Mv_1 + mu_1 = 0, \quad u_1 = \frac{M}{m}v_1.$$

The decrease in gas internal energy becomes kinetic energy of the piston and container:

$$\frac{1}{2}Mv_1^2 + \frac{1}{2}mu_1^2 = \frac{3}{2}nR(T - T_1).$$

Substitution yields

$$\frac{M(M + m)}{2m}v_1^2 = \frac{3}{2}nRT(1 - 2^{-2/3}),$$

hence

$$v_1 = \sqrt{\frac{3mnRT}{M(M + m)}(1 - 2^{-2/3})}.$$

After leaving, the piston experiences no further force.

Recoil during molecular escape.

For this stage, use the following molecular idealization: molecules move freely during evacuation; reflections from the cylindrical side wall preserve their axial velocity component; reflection from the closed end reverses that component. Changes in molecular speeds caused by the container's motion are neglected.

A molecule initially travelling toward the opening escapes directly. A molecule initially travelling toward the closed end reflects and subsequently escapes. Thus, if its initial axial velocity is v_x , its final axial velocity is approximately $|v_x|$.

Let μ be the mass of one molecule and k_B Boltzmann's constant. The initial Maxwell distribution of axial velocities at temperature T_1 is

$$f(v_x) = \sqrt{\frac{\mu}{2\pi k_B T_1}} \exp\left(-\frac{\mu v_x^2}{2k_B T_1}\right).$$

The required average, counting every initially present molecule once, is

$$\begin{aligned} \langle |v_x| \rangle &= 2 \int_0^\infty v_x f(v_x) dv_x \\ &= \sqrt{\frac{2k_B T_1}{\pi \mu}} = \sqrt{\frac{2RT_1}{\pi M_0}}. \end{aligned}$$

The gas has total mass nM_0 . Its final axial momentum is therefore

$$P_g = nM_0 \langle |v_x| \rangle = n \sqrt{\frac{2M_0 RT_1}{\pi}},$$

directed toward the opening.

Conservation of momentum for the complete system now gives

$$-Mv_f + mu_1 + P_g = 0.$$

Since $mu_1 = Mv_1$,

$$v_f = v_1 + \frac{P_g}{M}.$$

Inserting $T_1 = T2^{-2/3}$ gives

$$v_f = \sqrt{\frac{3mnRT}{M(M+m)}(1 - 2^{-2/3})} + \frac{n}{M} \sqrt{\frac{2M_0 RT}{\pi 2^{2/3}}}.$$

Both contributions propel the container toward its closed end.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

Problem 2: Lamp fed through a rheostat from an accumulator

An electric lamp of resistance $R_0 = 2 \, \Omega$ working at nominal voltage $U_0 = 4.5 \, \text{V}$ is connected to accumulator of electromotive force $E = 6 \, \text{V}$ and negligible internal resistance.

1. The nominal voltage of the lamp is ensured as the lamp is connected potentiometrically to the accumulator using a rheostat with resistance R . What should be the resistance R and what is the maximal electric current $I_{\max}$, flowing in the rheostat, if the efficiency of the system must not be smaller than $\eta_0 = 0.6$?
2. What is the maximal possible efficiency η of the system and how the lamp can be connected to the rheostat in this case?

Official English text of the problem.

Answers

1. The total rheostat resistance must satisfy

$$R \geq R_{\min} = 8.53 \, \Omega.$$

The maximum current in a rheostat section under the specified nominal operation is

$$I_{\max} = 2.81 \, \text{A}.$$

At the limiting design, the rheostat sections are $8.00 \, \Omega$ in parallel with the lamp and $0.533 \, \Omega$ in series with that parallel combination.

2. The maximum efficiency is

$$\eta_{\max} = 0.75 = 75\%.$$

Connect the lamp **in series** with an active rheostat resistance

$$R_{\text{series}} = \frac{2}{3} \Omega \approx 0.667 \Omega,$$

using the slider and one end terminal; leave the other end terminal unconnected.

Solution

We consider steady direct current, neglect the resistance of connecting wires, and use the given lamp resistance at its nominal operating point. The system efficiency is the electrical power delivered to the lamp divided by the power supplied by the accumulator:

$$\eta = \frac{P_{\text{lamp}}}{P_{\text{source}}}.$$

1. Potentiometer connection

Connect the two ends of the rheostat across the accumulator and connect the lamp between the slider and the negative terminal. Denote the two rheostat sections by:

- R_s : between the positive terminal and the slider;
- R_p : between the slider and the negative terminal, in parallel with the lamp.

Thus,

$$R = R_s + R_p.$$

At nominal lamp voltage, Ohm's law gives the lamp current

$$I_0 = \frac{U_0}{R_0} = \frac{4.5}{2} = 2.25 \text{ A}.$$

The current in the parallel rheostat section is

$$I_p = \frac{U_0}{R_p}.$$

By Kirchhoff's current law, the current through the series section, also the accumulator current, is

$$I = I_0 + I_p.$$

It is therefore the larger of the currents in the two rheostat sections.

The lamp power is

$$P_{\text{lamp}} = U_0 I_0 = \frac{U_0^2}{R_0} = 10.125 \text{ W}.$$

Consequently,

$$\eta = \frac{U_0 I_0}{EI} \geq \eta_0 \implies I \leq \frac{U_0 I_0}{E \eta_0}.$$

The largest permitted rheostat current is therefore

$$I_{\max} = \frac{4.5 \times 2.25}{6 \times 0.6} = 2.8125 \text{ A} \approx 2.81 \text{ A}.$$

The voltage across R_s is $E - U_0$, while that across R_p is U_0 . Hence

$$R_s = \frac{E - U_0}{I}, \quad R_p = \frac{U_0}{I - I_0},$$

and

$$R = \frac{E - U_0}{I} + \frac{U_0}{I - I_0}.$$

Both terms decrease as I increases for $I > I_0$. Thus the smallest total resistance compatible with the efficiency requirement occurs at $I = I_{\max}$, where $\eta = \eta_0$.

At this limiting point,

$$I_p = 2.8125 - 2.25 = 0.5625 \text{ A},$$

so

$$R_p = \frac{4.5}{0.5625} = 8.00 \text{ } \Omega, \quad R_s = \frac{6 - 4.5}{2.8125} = 0.5333 \dots \text{ } \Omega.$$

Therefore,

$$R_{\min} = 8.00 + 0.5333 \dots = 8.53 \text{ } \Omega.$$

Larger total resistances also satisfy the requirement when the slider is adjusted to maintain U_0 ; they draw less shunt current and give greater efficiency.

2. Maximum possible efficiency

At nominal voltage, the lamp must receive the fixed current I_0 . With a passive resistive connection, the accumulator current cannot be smaller than I_0 ; any additional parallel branch draws extra current.

Therefore,

$$\eta = \frac{U_0 I_0}{E I_{\text{source}}} \leq \frac{U_0}{E}.$$

Equality is achieved when the accumulator supplies only the lamp current, which requires a series connection:

$$\eta_{\max} = \frac{4.5}{6} = 0.75 = 75\%.$$

The active rheostat section must drop the excess voltage $E - U_0 = 1.5 \text{ V}$ while carrying $I_0 = 2.25 \text{ A}$:

$$R_{\text{series}} = \frac{E - U_0}{I_0} = \frac{1.5}{2.25} = \frac{2}{3} \text{ } \Omega \approx 0.667 \text{ } \Omega.$$

Connect the accumulator, lamp, and this rheostat section in one series loop. Use one end terminal and the slider, leaving the other end terminal unconnected.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

Problem 3: Radio interferometer on the sea shore

A detector of radiowaves in a radioastronomical observatory is placed on the sea beach at height $h = 2$ m above the sea level. After the rise of a star, radiating electromagnetic waves of wavelength $\lambda = 21$ cm, above the horizon the detector registers series of alternating maxima and minima. The registered signal is proportional to the intensity of the detected waves. The detector registers waves with electric vector, vibrating in a direction parallel to the sea surface.

1. Determine the angle between the star and the horizon in the moment when the detector registers maxima and minima (in general form).
2. Does the signal decrease or increase just after the rise of the star?
3. Determine the signal ratio of the first maximum to the next minimum. At reflection of the electromagnetic wave on the water surface, the ratio of the intensities of the electric field of the reflected (E_r) and incident (E_i) wave follows the low:

$$\frac{E_r}{E_i} = \frac{n - \cos \varphi}{n + \cos \varphi},$$

where n is the refraction index and φ is the incident angle of the wave. For the surface "air-water" for $\lambda = 21$ cm, the refraction index $n = 9$.

4. Does the ratio of the intensities of consecutive maxima and minima increase or decrease with rising of the star?

Assume that the sea surface is flat.

Official English text of the problem.

Answers

1. Writing α for the star's elevation above the horizon, the usual interference-fringe positions are

$$\alpha_m^{\max} \simeq \arcsin\left(\frac{(2m+1)\lambda}{4h}\right), \quad \alpha_m^{\min} \simeq \arcsin\left(\frac{m\lambda}{2h}\right).$$

Here $m = 0, 1, 2, \dots$, restricted to arguments not exceeding 1. The first maximum is at approximately 1.50° , and the following minimum at 3.01° . The small correction from angle-dependent reflection is quantified below.

2. The signal **increases** immediately after the star rises, starting from a minimum at the horizon.
3. The signal ratio is

$$\frac{S_{\text{first maximum}}}{S_{\text{next minimum}}} \simeq 2.96 \times 10^4.$$

4. The ratio of successive maximum signals to minimum signals **decreases** as the star rises.

Solution

1. Treat the distant star as a steady source of plane waves, and the flat sea as a coherent specular reflector. Let α be the elevation of the star and I_0 the intensity of the direct wave alone.

The reflected path can be calculated by replacing the detector, at height h , with its mirror image at depth h . The separation between detector and image is $2h$; its projection along the direction toward the star gives the path difference

$$\Delta = 2h \sin \alpha.$$

The detected electric field is parallel to the sea and perpendicular to the plane of incidence. Reflection from air into the optically denser water reverses this electric field, introducing a phase change of π .

Let r denote the positive reflected-to-incident field-amplitude ratio, and define the propagation phase difference

$$\delta = \frac{2\pi\Delta}{\lambda} = \frac{4\pi h}{\lambda} \sin \alpha.$$

Including the reflection phase reversal, the resultant intensity is

$$\boxed{\frac{I}{I_0} = 1 + r^2 - 2r \cos \delta = (1 - r)^2 + 4r \sin^2 \left(\frac{\delta}{2} \right)}.$$

Treating r as locally constant across a fringe, maxima occur when $\cos \delta = -1$, whereas minima occur when $\cos \delta = 1$. Thus

$$2h \sin \alpha_m^{\max} = \left(m + \frac{1}{2} \right) \lambda, \quad 2h \sin \alpha_m^{\min} = m \lambda,$$

which gives the formulas in the Answers section. Numerically,

$$\sin \alpha_m^{\max} = 0.02625(2m + 1), \quad \sin \alpha_m^{\min} = 0.05250m.$$

The horizon, $m = 0$, is a minimum.

For completeness, the supplied angle-dependent reflection coefficient causes small shifts from these phase-alignment positions. Since the incidence angle is measured from the surface normal,

$$\varphi = \frac{\pi}{2} - \alpha, \quad \cos \varphi = \sin \alpha.$$

Set $s = \sin \alpha$ and $K = 4\pi h/\lambda$. Then

$$r(s) = \frac{n - s}{n + s}, \quad r'(s) = -\frac{2n}{(n + s)^2}.$$

Differentiating the full intensity gives the exact condition for interior extrema:

$$\boxed{r'(s)[r(s) - \cos(Ks)] + Kr(s) \sin(Ks) = 0}.$$

The phase-alignment formulas give 1.50419° and 3.00941° for the first maximum and following minimum. Keeping the variation of r gives 1.50241° and 3.00940° , respectively. Thus the simple formulas give both angles correctly at the precision reported above.

2. At the horizon,

$$\alpha = 0, \quad \delta = 0, \quad r = 1.$$

The direct and reflected fields therefore cancel:

$$I(0) = 0.$$

As the star rises, the propagation phase difference becomes nonzero and the cancellation is lost. More explicitly, for small positive α , measured in radians,

$$r = 1 - \frac{2\alpha}{n} + O(\alpha^2), \quad \delta = K\alpha + O(\alpha^3).$$

Consequently,

$$\frac{I}{I_0} = \left(K^2 + \frac{4}{n^2} \right) \alpha^2 + O(\alpha^3).$$

The leading coefficient is positive, so the signal initially **increases quadratically with elevation**.

3. The given ratio E_r/E_i is a ratio of electric-field amplitudes. The measured signal is proportional to the square of the resultant field.

At the first maximum and following minimum, respectively,

$$s_1 = \frac{\lambda}{4h} = 0.02625, \quad s_2 = \frac{\lambda}{2h} = 0.05250.$$

Their reflection coefficients are different:

$$r_1 = \frac{9 - 0.02625}{9 + 0.02625} \simeq 0.994184,$$

$$r_2 = \frac{9 - 0.05250}{9 + 0.05250} \simeq 0.988401.$$

At a maximum the field amplitudes add, while at a minimum they subtract. Hence

$$I_{\max,1} = I_0(1 + r_1)^2, \quad I_{\min,1} = I_0(1 - r_2)^2.$$

Using

$$1 + r_1 = \frac{2n}{n + s_1}, \quad 1 - r_2 = \frac{2s_2}{n + s_2},$$

the requested ratio becomes

$$R = \left[\frac{n(n + s_2)}{s_2(n + s_1)} \right]^2$$

$$= \left[\frac{9(9 + 0.05250)}{0.05250(9 + 0.02625)} \right]^2$$

$$\simeq 29558.9.$$

Therefore,

$$\boxed{R \simeq 2.96 \times 10^4}.$$

Using the strict extrema from part 1 changes this ratio by less than 10^{-5} relatively, leaving the reported result unchanged.

4. As the star rises, $\sin \alpha$ increases and the reflection amplitude decreases:

$$\frac{dr}{d\alpha} = -\frac{2n \cos \alpha}{(n + \sin \alpha)^2} < 0 \quad \left(0 < \alpha < \frac{\pi}{2} \right).$$

The constructive-interference envelope is

$$I_{\text{upper}} = I_0(1 + r)^2,$$

which decreases as r decreases. The destructive-interference envelope is

$$I_{\text{lower}} = I_0(1 - r)^2,$$

which increases. Thus later maxima become weaker while later minima become stronger, so the maximum-to-minimum signal ratio **decreases**.

The solution was checked against the official solution.