

IPhO 1979

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Problem 1: Landing a spacecraft from a lunar orbit

A space rocket with mass $M = 12\text{t}$ is moving around the Moon along the circular orbit at the height of $h = 100\text{ km}$. The engine is activated for a short time to pass at the lunar landing orbit. The velocity of the ejected gases $u = 10^4\text{ m/s}$. The Moon radius $R_M = 1,7 \cdot 10^3\text{ km}$, the acceleration of gravity near the Moon surface $g_M = 1.7\text{ m/s}^2$

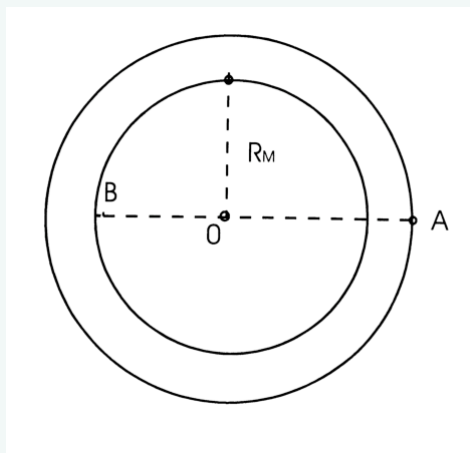


Fig.1

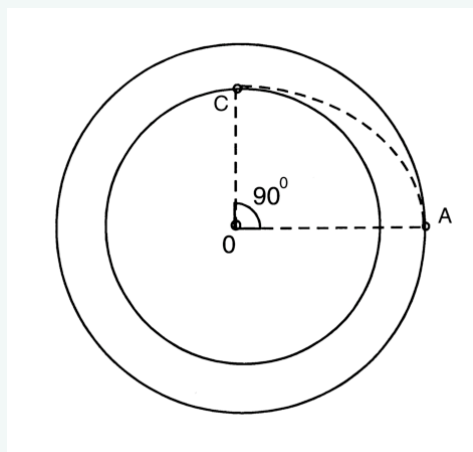


Fig.2

- 1). What amount of fuel should be spent so that when activating the braking engine at point A of the trajectory, the rocket would land on the Moon at point B (Fig.1)?
- 2). In the second scenario of landing, at point A the rocket is given an impulse directed towards the center of the Moon, to put the rocket to the orbit meeting the Moon surface at point C (Fig.2). What amount of fuel is needed in this case?

Answers

- 1). $\Delta m_1 \simeq 29 \text{ kg}$, for a tangential braking velocity change of 23.77 m/s .
- 2). $\Delta m_2 \simeq 1.2 \times 10^2 \text{ kg}$, for an inward radial velocity change of 97.18 m/s .

The ideal rocket equation gives approximately 28.49 kg and 116.05 kg , respectively, before rounding to the precision of the supplied data.

Solution

Treat the Moon as spherical, with an inverse-square gravitational field. The engine operates briefly enough that the spacecraft's position and the gravitational impulse during the burn can be neglected. The exhaust speed u is measured relative to the spacecraft.

Write

$$R = R_M = 1.7 \times 10^6 \text{ m}, \quad r_0 = R + h = 1.8 \times 10^6 \text{ m}.$$

The Moon's gravitational parameter is

$$\mu = GM_{\text{Moon}} = g_M R^2 = 4.913 \times 10^{12} \text{ m}^3/\text{s}^2.$$

Thus the initial circular speed is

$$v_0 = \sqrt{\frac{\mu}{r_0}} \simeq 1652.10 \text{ m/s}.$$

For a burn producing a velocity change of magnitude Δv , conservation of momentum gives the ideal rocket equation:

$$\Delta v = u \ln \frac{M}{M - \Delta m}.$$

Consequently,

$$\Delta m = M \left(1 - e^{-\Delta v/u} \right).$$

When $\Delta v/u \ll 1$, this becomes

$$\Delta m \simeq \frac{M \Delta v}{u}.$$

Here, reaching the surface is understood as the requested orbital contact. The spacecraft retains a nonzero tangential velocity there; a soft landing would require further braking.

1). Tangential braking at A, reaching B

The braking impulse acts opposite the initial velocity. Immediately afterward, the velocity remains tangential but is smaller than the circular speed. Therefore A is the apocentre of the new elliptical orbit. Its diametrically opposite pericentre must be B, at radius R .

Let v_A and v_B be the speeds on this ellipse at A and B. Conservation of angular momentum per unit mass gives

$$r_0 v_A = R v_B.$$

Conservation of mechanical energy per unit mass gives

$$\frac{v_A^2}{2} - \frac{\mu}{r_0} = \frac{v_B^2}{2} - \frac{\mu}{R}.$$

Eliminating v_B ,

$$v_A^2 \left(\frac{r_0^2}{R^2} - 1 \right) = 2\mu \left(\frac{1}{R} - \frac{1}{r_0} \right),$$

so

$$v_A^2 = \frac{2\mu R}{r_0(r_0 + R)}.$$

The required reduction in velocity is therefore

$$\Delta v_1 = v_0 - v_A = \sqrt{\frac{\mu}{r_0}} \left(1 - \sqrt{\frac{2R}{r_0 + R}} \right).$$

Numerically,

$$\Delta v_1 = 1652.10 \left(1 - \sqrt{\frac{3.4}{3.5}} \right) \simeq 23.77 \text{ m/s}.$$

Hence

$$\Delta m_1 = 12000 \left(1 - e^{-23.7725/10000} \right) \text{ kg} \simeq 28.49 \text{ kg},$$

or, to sensible significant figures,

$$\Delta m_1 \simeq 29 \text{ kg}.$$

The radius decreases from r_0 to R along the transfer, so B is the first point where the orbit touches the lunar surface.

2). Radial impulse at A, reaching C

Let the inward velocity imparted by the engine be w . Immediately after the burn,

$$v_r = -w, \quad v_t = v_0.$$

The velocity change has magnitude

$$\Delta v_2 = w.$$

Because the impulse is radial, the specific angular momentum is unchanged:

$$\ell = r_0 v_0, \quad \ell^2 = \mu r_0.$$

Let θ be the angle swept around the Moon from A in the direction of motion. The required contact point C has

$$\theta_C = \frac{\pi}{2}.$$

To determine the orbit, put $q(\theta) = 1/r(\theta)$. Angular momentum conservation gives

$$\dot{r} = -\ell q',$$

where primes denote derivatives with respect to θ . Newton's radial equation for inverse-square gravity then yields

$$q'' + q = \frac{\mu}{\ell^2} = \frac{1}{r_0}.$$

At A,

$$q(0) = \frac{1}{r_0}, \quad q'(0) = \frac{w}{\ell}.$$

The solution satisfying these conditions is

$$q(\theta) = \frac{1}{r_0} + \frac{w}{\ell} \sin \theta,$$

or

$$r(\theta) = \frac{r_0}{1 + (w/v_0) \sin \theta}.$$

Imposing $r(\pi/2) = R$ gives

$$R = \frac{r_0}{1 + w/v_0},$$

and therefore

$$\Delta v_2 = w = v_0 \left(\frac{r_0}{R} - 1 \right) = v_0 \frac{h}{R} = h \sqrt{\frac{g_M}{R + h}}.$$

The radius decreases continuously between A and C, and its derivative vanishes at C. Thus C is the pericentre and the first contact with the lunar surface.

Numerically,

$$\Delta v_2 = 1652.10 \frac{100}{1700} \simeq 97.18 \text{ m/s}.$$

The required fuel is

$$\Delta m_2 = 12000 \left(1 - e^{-97.1825/10000} \right) \text{ kg} \simeq 116.05 \text{ kg},$$

hence

$$\Delta m_2 \simeq 1.2 \times 10^2 \text{ kg}.$$

Using the small-fuel approximation gives 28.53 kg for part 1 and 116.62 kg for part 2, consistent with the rounded answers.

The solution was checked against the official solution.

Problem 2: Weighing a sample in dry and in humid air

Brass weights are used to weigh an aluminum-made sample on an analytical balance. The weighing is ones in dry air and another time in humid air with the water vapor pressure $P_h = 2 \cdot 10^3$ Pa. The total atmospheric pressure ($P = 10^5$ Pa) and the temperature ($t = 20^\circ$ C) are the same in both cases.

What should the mass of the sample be to be able to tell the difference in the balance readings provided their sensitivity is $m_0 = 0.1$ mg ?

Aluminum density $\rho_1 = 2700$ kg/m³, brass density $\rho_2 = 8500$ kg/m³.

Official English text of the problem.

Answers

- **Problem 2.** The sample must have a mass of at least approximately

$$m_{\min} \simeq 4.4 \times 10^{-2} \text{ kg} = 44 \text{ g.}$$

The balance reading is higher in humid air; at the threshold, the increase is 0.1 mg.

Solution

Problem 2.

Treat dry air and water vapor as ideal gases, and assume that the masses and volumes of the sample and weights remain unchanged. We use the brass density 8500 kg/m³, interpreting the preceding dot in the statement as a printing error.

First find the change in air density. For an ideal gas of molar mass M ,

$$\rho = \frac{pM}{RT}.$$

In humid air, Dalton's law gives the dry-air partial pressure as $P - P_h$. Thus, writing M_a and M_v for the molar masses of dry air and water vapor,

$$\rho_d = \frac{PM_a}{RT}, \quad \rho_h = \frac{(P - P_h)M_a + P_h M_v}{RT}.$$

Consequently,

$$\Delta\rho \equiv \rho_d - \rho_h = \frac{P_h(M_a - M_v)}{RT}.$$

Water vapor has a smaller molar mass than dry air, so humid air is less dense at the same total pressure and temperature.

Using the standard approximate constants

$$M_a = 29 \times 10^{-3} \text{ kg/mol}, \quad M_v = 18 \times 10^{-3} \text{ kg/mol}, \quad R = 8.314 \text{ J/(mol K)},$$

and $T = 293.15$ K, we obtain

$$\Delta\rho = \frac{(2 \times 10^3)(29 - 18) \times 10^{-3}}{8.314 \times 293.15} \simeq 9.03 \times 10^{-3} \text{ kg/m}^3.$$

Next account for buoyancy on **both** the sample and the brass weights. Let m be the true sample mass and m_b the total brass mass required for balance in air of density ρ . Their volumes are m/ρ_1 and m_b/ρ_2 . Archimedes' principle and equality of the apparent weights give

$$mg - \rho g \frac{m}{\rho_1} = m_b g - \rho g \frac{m_b}{\rho_2}.$$

Therefore, the balance reading is

$$m_b(\rho) = m \frac{1 - \rho/\rho_1}{1 - \rho/\rho_2}.$$

The difference between the humid-air and dry-air readings is

$$\Delta m_b = \frac{m(\rho_d - \rho_h) \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right)}{\left(1 - \frac{\rho_d}{\rho_2} \right) \left(1 - \frac{\rho_h}{\rho_2} \right)}.$$

Here $\rho_d \simeq 1.190 \text{ kg/m}^3$ and $\rho_h \simeq 1.181 \text{ kg/m}^3$, so the denominator differs from unity by only about 0.028%. To the required accuracy,

$$\Delta m_b \simeq m \Delta\rho \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \simeq m(2.28 \times 10^{-6}).$$

This difference is positive: the aluminum sample has a greater volume per unit mass, so reducing air density increases its apparent weight more than that of the brass weights.

For a detectable difference, require

$$\Delta m_b \geq m_0 = 0.1 \text{ mg} = 1.0 \times 10^{-7} \text{ kg}.$$

Hence

$$m_{\min} \simeq \frac{1.0 \times 10^{-7}}{(9.03 \times 10^{-3}) \left(\frac{1}{2700} - \frac{1}{8500} \right)} \simeq 4.38 \times 10^{-2} \text{ kg} \simeq \boxed{44 \text{ g}}.$$

The solution was checked against the official solution.

Problem 3: Laser location of the Moon

During the Soviet-French experiment on the optical location of the Moon the light pulse of a ruby laser ($\lambda = 0,69 \mu\text{m}$) was directed to the Moon's surface by the telescope with a diameter of the mirror $D = 2,6 \text{ m}$. The reflector on the Moon's surface reflected the light backward as an ideal mirror with the diameter $d = 20 \text{ cm}$. The reflected light was then collected by the same telescope and focused at the photodetector.

- 1) What must the accuracy to direct the telescope optical axis be in this experiment?
- 2) What part of emitted laser energy can be detected after reflection on the Moon, if we neglect the light losses in the Earth's atmosphere?

3) Can we see a reflected light pulse with naked eye if the energy of single laser pulse $E = 1 \text{ J}$ and the threshold sensitivity of eye is equal $n = 100$ light quantum?

4) Suppose the Moon's surface reflects $\alpha = 10\%$ of the incident light in the spatial angle 2π steradian, estimate the advantage of a using reflector.

The distance from the Earth to the Moon is $L = 380000 \text{ km}$. The diameter of pupil of the eye is $d_p = 5 \text{ mm}$. Plank constant is $h = 6.6 \cdot 10^{-34} \text{ J} \cdot \text{s}$.

Official English text of the problem.

Answers

The numerical results below assume uniform coherent illumination of the circular telescope aperture, a centered reflector, and ideal collection.

1. Pointing accuracy of order

$$\delta\theta \lesssim 3 \times 10^{-7} \text{ rad} \approx 0.06''.$$

2. The fraction collected by the telescope is

$$\eta = \left(\frac{\pi D d}{4 \lambda L} \right)^4 \approx 5.9 \times 10^{-12}.$$

3. **No.** The unaided eye receives approximately

$$N_{\text{eye}} \approx 76 \text{ photons} < 100.$$

4. Relative to diffuse reflection from the entire illuminated lunar patch, the reflector increases the received energy by approximately

$$G \approx 1.0 \times 10^7.$$

The elementary approximation of uniformly illuminated diffraction spots gives $\eta \approx 10^{-12}$, $N_{\text{eye}} \approx 12$, and $G \approx 1.7 \times 10^6$. The distinction comes from the diffraction intensity profile, as explained below.

Solution

We take

$$\lambda = 6.9 \times 10^{-7} \text{ m}, \quad L = 3.8 \times 10^8 \text{ m}, \quad c = 3.0 \times 10^8 \text{ m s}^{-1}.$$

We neglect atmospheric absorption, optical losses, and detector inefficiency. The laser fills the telescope aperture coherently, and the reflector is centered in the outgoing beam.

1) Required pointing accuracy

A finite aperture produces diffraction. For an aperture of diameter a , the characteristic angular spread is

$$\theta \sim \frac{\lambda}{a}.$$

For a uniformly illuminated circular aperture, the first dark ring occurs at

$$\theta_1 = 1.22 \frac{\lambda}{a}.$$

For the transmitting telescope,

$$\frac{\lambda}{D} = \frac{6.9 \times 10^{-7}}{2.6} = 2.65 \times 10^{-7} \text{ rad} = 0.0547''.$$

The first-dark-ring angle is $3.24 \times 10^{-7} \text{ rad} = 0.0668''$.

The corresponding central diffraction spot on the Moon has radius

$$r_1 = L\theta_1 \approx 123 \text{ m},$$

much larger than the reflector's radius 0.10 m.

A pointing error $\delta\theta$ moves the beam center by $L\delta\theta$. To keep the reflector in the central diffraction spot, this displacement must be smaller than roughly 10^2 m . Thus

$$\boxed{\delta\theta \lesssim 3 \times 10^{-7} \text{ rad} \approx 0.06''}.$$

For nearly maximal return, the pointing error should be appreciably smaller than this characteristic width.

2) Fraction of laser energy collected after reflection

Diffraction occurs twice: first at the transmitting telescope and then at the lunar reflector.

To retain the circular-aperture numerical factor, consider a uniformly illuminated aperture of area A . In the far field, the contributions from all parts of the aperture arrive in phase on its axis. The Huygens–Fresnel formula gives the axial field amplitude, apart from an overall phase, as

$$u(0, L) = \frac{u_0 A}{i\lambda L}.$$

Because the transmitted energy is proportional to $|u_0|^2 A$, the axial energy per unit area, or fluence, is

$$\boxed{\mathcal{F}(0, L) = \frac{EA}{\lambda^2 L^2}}.$$

Consequently, a small receiving aperture of area B , centered on the axis, intercepts the fraction

$$\frac{E_B}{E} \approx \frac{AB}{\lambda^2 L^2}.$$

Here the telescope and reflector areas are

$$A_D = \frac{\pi D^2}{4} = 5.31 \text{ m}^2, \quad A_d = \frac{\pi d^2}{4} = 3.14 \times 10^{-2} \text{ m}^2.$$

The reflector is tiny compared with the outgoing diffraction spot, so the incident fluence is essentially constant across it. The fraction reaching it is therefore

$$f_{\text{out}} = \frac{A_D A_d}{\lambda^2 L^2} = 2.43 \times 10^{-6}.$$

The illuminated reflector acts as a coherent aperture of diameter d . Its returning diffraction pattern has first-dark-ring radius

$$r_{\text{return}} = 1.22 \frac{\lambda L}{d} \approx 1.60 \times 10^3 \text{ m}.$$

The telescope is tiny compared with this pattern, so the same central-fluence calculation applies:

$$f_{\text{back}} = \frac{A_d A_D}{\lambda^2 L^2} = 2.43 \times 10^{-6}.$$

Multiplying the two fractions,

$$\eta = f_{\text{out}} f_{\text{back}} = \left(\frac{A_D A_d}{\lambda^2 L^2} \right)^2 = \left(\frac{\pi D d}{4 \lambda L} \right)^4.$$

Thus

$$\boxed{\eta = 5.9 \times 10^{-12}}.$$

3) Visibility to the naked eye

The returning fluence is essentially uniform over both the telescope aperture and the much smaller pupil. The energy entering an unaided eye is therefore

$$E_{\text{eye}} = E \eta \frac{A_p}{A_D} = E \eta \left(\frac{d_p}{D} \right)^2.$$

Substitution gives

$$E_{\text{eye}} = (1)(5.89 \times 10^{-12}) \left(\frac{0.005}{2.6} \right)^2 = 2.18 \times 10^{-17} \text{ J}.$$

The energy of one photon is

$$\varepsilon_\gamma = \frac{hc}{\lambda} = \frac{(6.6 \times 10^{-34})(3.0 \times 10^8)}{6.9 \times 10^{-7}} = 2.87 \times 10^{-19} \text{ J}.$$

Hence

$$N_{\text{eye}} = \frac{E_{\text{eye}}}{\varepsilon_\gamma} \approx 75.9.$$

Since

$$\boxed{N_{\text{eye}} \approx 76 < 100},$$

the pulse does not meet the stated naked-eye visibility threshold, even under these ideal assumptions.

Uniform, in-phase illumination maximizes the axial fluence for a fixed aperture and transmitted energy. Thus an illumination profile giving a lower central intensity would not improve this conclusion.

4) Advantage over reflection from the lunar surface

For the requested estimate, suppose the diffusely reflected energy αE is distributed uniformly over 2π steradians. The receiving telescope subtends the solid angle

$$\Omega_D \approx \frac{A_D}{L^2}.$$

The fraction of the emitted energy collected from the diffuse surface is

$$\eta_{\text{diffuse}} = \alpha \frac{\Omega_D}{2\pi} = \frac{\alpha A_D}{2\pi L^2} = \frac{\alpha D^2}{8 L^2}.$$

Numerically,

$$\eta_{\text{diffuse}} = \frac{0.10(2.6)^2}{8(3.8 \times 10^8)^2} = 5.85 \times 10^{-19}.$$

This includes reflection from the **whole illuminated lunar patch**.

The reflector's advantage is therefore

$$G = \frac{\eta}{\eta_{\text{diffuse}}} = \frac{5.89 \times 10^{-12}}{5.85 \times 10^{-19}} \approx 1.01 \times 10^7.$$

Thus

$$\boxed{G \approx 10^7}.$$

Elementary diffraction-spot approximation

A simpler estimate treats each beam as uniformly spread over a disk of radius

$$r \approx \frac{\lambda L}{a},$$

where a is the emitting aperture diameter. It gives

$$r_{\text{Moon}} \approx 101 \text{ m}, \quad r_{\text{Earth}} \approx 1311 \text{ m}.$$

Taking the intercepted energy fractions as area ratios then yields

$$\eta_{\text{spot}} \approx \frac{(d/2)^2}{r_{\text{Moon}}^2} \frac{(D/2)^2}{r_{\text{Earth}}^2} = \left(\frac{Dd}{2\lambda L} \right)^4 = 9.7 \times 10^{-13}.$$

The corresponding results are

$$N_{\text{eye,spot}} \approx 12, \quad G_{\text{spot}} \approx 1.7 \times 10^6.$$

This is a useful elementary estimate. The circular-aperture calculation gives a larger return because the actual diffraction pattern is centrally peaked, and both receiving apertures sample its central region.

The solution was checked against the official solution.