

IPhO 1977

Theory problems and solutions · Hradec Králové, Czechoslovakia · 3 theory problems with complete solutions

eulersolve.org/ipho/1977/

Problem 1: Otto cycle of a four-stroke combustion engine

The compression ratio of a four-stroke internal combustion engine is $\varepsilon = 9.5$. The engine draws in air and gaseous fuel at a temperature 27°C at a pressure $1\text{ atm} = 100\text{ kPa}$. Compression follows an adiabatic process from point 1 to point 2, see Fig. 1. The pressure in the cylinder is doubled during the mixture ignition (2–3). The hot exhaust gas expands adiabatically to the volume V_2 pushing the piston downwards (3–4). Then the exhaust valve opens and the pressure gets back to the initial value of 1 atm . All processes in the cylinder are supposed to be ideal. The Poisson constant (i.e. the ratio of specific heats C_p/C_V) for the mixture and exhaust gas is $\kappa = 1.40$. (The compression ratio is the ratio of the volume of the cylinder when the piston is at the bottom to the volume when the piston is at the top.)

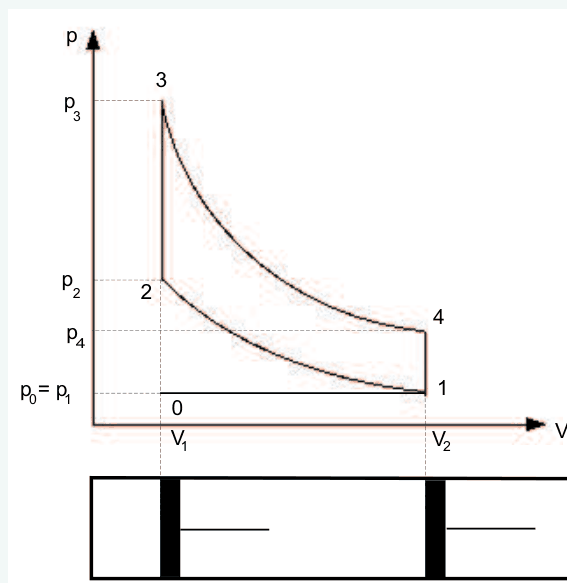


Figure 1:

- Which processes run between the points 0–1, 2–3, 4–1, 1–0?
- Determine the pressure and the temperature in the states 1, 2, 3 and 4.
- Find the thermal efficiency of the cycle.
- Discuss obtained results. Are they realistic?

Official English text of the problem.

Answers

- **a)** $0 \rightarrow 1$: isobaric intake; $2 \rightarrow 3$: constant-volume combustion and heat addition; $4 \rightarrow 1$: constant-volume pressure decrease as the exhaust valve opens, represented by heat rejection in the ideal Otto cycle; $1 \rightarrow 0$: isobaric exhaust.

- **b)** Using $27^\circ\text{C} \simeq 300\text{ K}$:

State	Pressure	Temperature
1	100 kPa	300 K
2	$2.34 \times 10^3\text{ kPa}$	738 K
3	$4.68 \times 10^3\text{ kPa}$	$1.48 \times 10^3\text{ K}$
4	200 kPa	600 K

- **c)** $\eta = 1 - \varepsilon^{1-\kappa} = 0.594 \approx 59.4\%$.
- **d)** The pressures and temperatures are plausible orders of magnitude for a combustion engine. The efficiency is an optimistic ideal value; heat transfer, variable heat capacities, finite combustion and gas-exchange losses reduce actual efficiency, and friction further reduces shaft output.

Solution

a) Identification of the processes

Let the minimum and maximum cylinder volumes be

$$V_{\min} = V_1, \quad V_{\max} = V_2, \quad \frac{V_{\max}}{V_{\min}} = \varepsilon = 9.5,$$

where V_1, V_2 are the volume labels in the figure. States 2 and 3 occur at $V_{\min}$; states 1 and 4 occur at $V_{\max}$.

- **$0 \rightarrow 1$: Isobaric intake.** The piston moves downwards, drawing in fresh air and gaseous fuel at the external pressure $p_0 = p_1$.
- **$2 \rightarrow 3$: Isochoric combustion.** Idealized instantaneous combustion raises the pressure and temperature while the piston remains at its upper position. The gas does no boundary work because its volume is constant.
- **$4 \rightarrow 1$: Isochoric pressure decrease.** The exhaust valve opens at the lower piston position, and gas escapes until the cylinder pressure reaches the external pressure. This rapid exhaust release is represented by constant-volume heat rejection in the ideal closed Otto cycle.
- **$1 \rightarrow 0$: Isobaric exhaust.** The piston moves upwards, expelling exhaust gas at the external pressure.

Intake and exhaust involve matter crossing the cylinder boundary. Their idealized pressure–volume paths coincide.

b) Pressures and temperatures

Treat the working gas as ideal, with constant heat capacities and $\kappa = C_p/C_V = 1.40$. During each ideal adiabatic stroke,

$$pV^\kappa = \text{constant}, \quad TV^{\kappa-1} = \text{constant}.$$

These relations follow from the first law,

$$nC_V dT = -p dV,$$

together with $pV = nRT$ and $R/C_V = \kappa - 1$, where C_V is the molar heat capacity.

At **state 1**, the supplied conditions give

$$p_1 = 100 \text{ kPa}, \quad T_1 = 27^\circ\text{C} \simeq 300 \text{ K}.$$

During **adiabatic compression** $1 \rightarrow 2$,

$$p_2 = p_1 \left(\frac{V_{\max}}{V_{\min}} \right)^\kappa = p_1 \varepsilon^\kappa,$$
$$T_2 = T_1 \left(\frac{V_{\max}}{V_{\min}} \right)^{\kappa-1} = T_1 \varepsilon^{\kappa-1}.$$

Numerically,

$$\varepsilon^{\kappa-1} = 9.5^{0.40} = 2.46087, \quad \varepsilon^\kappa = 9.5^{1.40} = 23.3783,$$

so

$$\boxed{p_2 \simeq 2.34 \times 10^3 \text{ kPa}}, \quad \boxed{T_2 \simeq 738 \text{ K}}.$$

During **constant-volume ignition** $2 \rightarrow 3$, the pressure doubles. Under the ideal working-gas approximation, p/T is constant at fixed volume, hence

$$p_3 = 2p_2, \quad T_3 = 2T_2.$$

Thus

$$\boxed{p_3 \simeq 4.68 \times 10^3 \text{ kPa}}, \quad \boxed{T_3 \simeq 1.48 \times 10^3 \text{ K}}.$$

During **adiabatic expansion** $3 \rightarrow 4$,

$$p_4 = p_3 \left(\frac{V_{\min}}{V_{\max}} \right)^\kappa = \frac{2p_1 \varepsilon^\kappa}{\varepsilon^\kappa} = 2p_1,$$
$$T_4 = T_3 \left(\frac{V_{\min}}{V_{\max}} \right)^{\kappa-1} = \frac{2T_1 \varepsilon^{\kappa-1}}{\varepsilon^{\kappa-1}} = 2T_1.$$

Therefore,

$$\boxed{p_4 = 200 \text{ kPa}}, \quad \boxed{T_4 \simeq 600 \text{ K}}.$$

c) Thermal efficiency

Take work done by the gas as positive. The intake and exhaust strokes contribute equal and opposite work:

$$W_{01} = p_1(V_{\max} - V_{\min}), \quad W_{10} = -p_1(V_{\max} - V_{\min}).$$

They therefore cancel. The constant-volume processes contribute no boundary work.

For the two adiabatic strokes, the first law gives

$$W_{12} = nC_V(T_1 - T_2), \quad W_{34} = nC_V(T_3 - T_4).$$

The net work per cycle is consequently

$$W_{\text{net}} = nC_V[(T_3 - T_2) - (T_4 - T_1)].$$

The combustion energy is represented by heat supplied at constant volume:

$$Q_{\text{in}} = nC_V(T_3 - T_2).$$

Hence

$$\eta = \frac{W_{\text{net}}}{Q_{\text{in}}} = 1 - \frac{T_4 - T_1}{T_3 - T_2}.$$

The quantity $nC_V(T_4 - T_1)$ is the equivalent heat rejected in the closed Otto-cycle representation.

Since $T_3 = 2T_2$ and $T_4 = 2T_1$,

$$\eta = 1 - \frac{T_1}{T_2} = 1 - \frac{1}{\varepsilon^{\kappa-1}}.$$

Therefore,

$$\eta = 1 - 9.5^{-0.40} = 0.59364 \approx 59.4\%.$$

Within this ideal model, efficiency depends on the compression ratio and κ . The specified pressure doubling determines the temperatures and work per charge, but cancels from the efficiency.

d) Discussion of realism

The compression temperature is approximately 465°C , the peak temperature approximately 1200°C , and the peak pressure approximately 4.68 MPa. These are reasonable orders of magnitude for a combustion engine.

At the end of expansion, the gas remains at approximately 327°C and twice atmospheric pressure. Thus the exhaust still carries substantial energy. Even the ideal cycle rejects

$$1 - \eta \simeq 40.6\%$$

of the supplied energy.

An actual engine departs from the model in several ways:

- Heat passes from the hot gas into the cylinder walls, so compression and expansion are not perfectly adiabatic.
- Combustion takes finite time while the piston moves, so heat addition is not exactly isochoric.
- Heat capacities depend on temperature and composition. Their increase at high temperature makes a constant $\kappa = 1.40$ an optimistic approximation.
- Intake and exhaust require pressure differences across valves, producing pumping losses; their work contributions no longer cancel exactly.
- Incomplete combustion and mechanical friction reduce useful output.

Moreover, actual exhaust release is an open process: escaping gas carries energy out of the cylinder, and intake supplies a fresh charge. Constant-volume cooling of a fixed gas charge is its ideal Otto-cycle representation.

Thus the calculated 59.4% is the efficiency of the stipulated ideal model; the actual efficiency and shaft output will be lower.

The solution was checked against the official solution.

Problem 2: Colours of a soap film

Dipping the frame in a soap solution, the soap forms a rectangle film of length b and height h . White light falls on the film at an angle α (measured with respect to the normal direction). The reflected light displays a green color of wavelength λ_0 .

- Find out if it is possible to determine the mass of the soap film using the laboratory scales which has calibration accuracy of 0.1 mg.
- What color does the thinnest possible soap film display being seen from the perpendicular direction? Derive the related equations.

Constants and given data: relative refractive index $n = 1.33$, the wavelength of the reflected green light $\lambda_0 = 500 \text{ nm}$, $\alpha = 30^\circ$, $b = 0.020 \text{ m}$, $h = 0.030 \text{ m}$, density $\rho = 1000 \text{ kg m}^{-3}$.

Official English text of the problem.

Answers

- a)** The permitted thicknesses and masses are

$$d_j = (2j + 1) \frac{\lambda_0}{4\sqrt{n^2 - \sin^2 \alpha}}, \quad M_j = (2j + 1)(0.0609 \text{ mg}), \quad j = 0, 1, 2, \dots$$

The thinnest film has $d_{\min} \simeq 0.10 \text{ }\mu\text{m}$ and $M_{\min} \simeq 0.061 \text{ mg}$, so its mass cannot be determined reliably with the stated balance. Higher orders give larger masses; without specifying the order, there is no unconditional yes-or-no answer for an arbitrary green film.

- b)** With illumination remaining at 30° , **there is no specularly reflected light along the normal**, hence no reflected interference colour in that direction. The ideal film is dark in reflected light. A 540 nm green maximum would arise if the illumination were also normal, as derived below.

Solution

Treat the film as uniform, plane-parallel, and surrounded by air. Neglect absorption, refractive-index dispersion, and edge diffraction. Interpret the stated green wavelength as a maximum of the reflected intensity.

- Let d be the film thickness and r the refracted angle inside it. Snell's law gives

$$n \sin r = \sin \alpha, \quad r = \arcsin\left(\frac{\sin 30^\circ}{1.33}\right) \simeq 22.1^\circ.$$

The reflected light contains a contribution from the front surface and another that enters the film, reflects from the rear surface, and emerges. Comparing their paths at a common outgoing wavefront, their geometrical optical-path difference is

$$\Delta = \frac{2nd}{\cos r} - 2d \tan r \sin \alpha.$$

Using $\sin \alpha = n \sin r$,

$$\Delta = \frac{2nd(1 - \sin^2 r)}{\cos r} = 2nd \cos r.$$

Reflection at the front air–soap interface introduces a phase reversal of π ; reflection at the rear soap–air interface does not. Consequently, constructive interference in reflection requires

$$\frac{2\pi}{\lambda} 2nd \cos r + \pi = 2\pi(j + 1),$$

or

$$\boxed{2nd \cos r = \left(j + \frac{1}{2}\right)\lambda}, \quad j = 0, 1, 2, \dots$$

For the observed maximum at λ_0 ,

$$d_j = \frac{(2j + 1)\lambda_0}{4n \cos r} = \frac{(2j + 1)\lambda_0}{4\sqrt{n^2 - \sin^2 \alpha}}.$$

The minimum thickness corresponds to $j = 0$:

$$d_{\min} = \frac{500 \times 10^{-9}}{4\sqrt{1.33^2 - \sin^2 30^\circ}} = 1.014 \times 10^{-7} \text{ m} \simeq 0.10 \text{ } \mu\text{m}.$$

The mass is density times volume:

$$M_j = \rho b h d_j.$$

Thus

$$M_{\min} = (1000)(0.020)(0.030)(1.014 \times 10^{-7}) = 6.09 \times 10^{-8} \text{ kg} = \boxed{0.061 \text{ mg}}.$$

This is smaller than the balance's 0.1 mg calibration accuracy. Therefore, the minimum-thickness film cannot be weighed reliably with that balance.

The interference order matters: the next allowed mass is

$$M_1 = 3M_{\min} \simeq 0.18 \text{ mg},$$

which exceeds the stated accuracy, although the fractional uncertainty would still be large. The green maximum alone therefore does not establish whether an arbitrary film is sufficiently massive for a useful weighing.

b) The incident direction stays fixed at 30° .

The front-surface reflected ray leaves at the same angle to the normal as the incident ray. For the ray reflected inside the film, reflection preserves its internal angle r , and refraction upon emergence gives

$$\sin \theta_{\text{out}} = n \sin r = \sin \alpha.$$

Hence

$$\theta_{\text{out}} = \alpha = 30^\circ.$$

All the interfering reflected rays therefore travel in the 30° specular direction. An observer looking along the normal, $\theta_{\text{out}} = 0$, receives none of these rays. Within the stated approximations,

$$I_{\text{reflected}}(0, \lambda) = 0.$$

There is consequently **no normal-view reflected interference colour**, regardless of which allowed thickness the film has. Additional illumination or scattering would require further information.

For completeness, the same minimum-thickness film under **normal illumination and normal observation** would satisfy

$$2nd_{\text{min}} = \frac{\lambda_{\perp}}{2}.$$

Its first reflected maximum would then be

$$\lambda_{\perp} = 4nd_{\text{min}} = \frac{\lambda_0}{\cos r} = \frac{n\lambda_0}{\sqrt{n^2 - \sin^2 \alpha}} \simeq 5.40 \times 10^{-7} \text{ m} = \boxed{540 \text{ nm}},$$

which is green. This value applies to that changed illumination geometry.

The solution was checked against the official solution.

Problem 3: Electron beam aimed at a target by a magnetic field

An electron gun T emits electrons accelerated by a potential difference U in a vacuum in the direction of the line a as shown in Fig. 2. The target M is placed at a distance d from the electron gun in such a way that the line segment connecting the points T and M and the line a subtend the angle α as shown in Fig. 2. Find the magnetic induction B of the uniform magnetic field

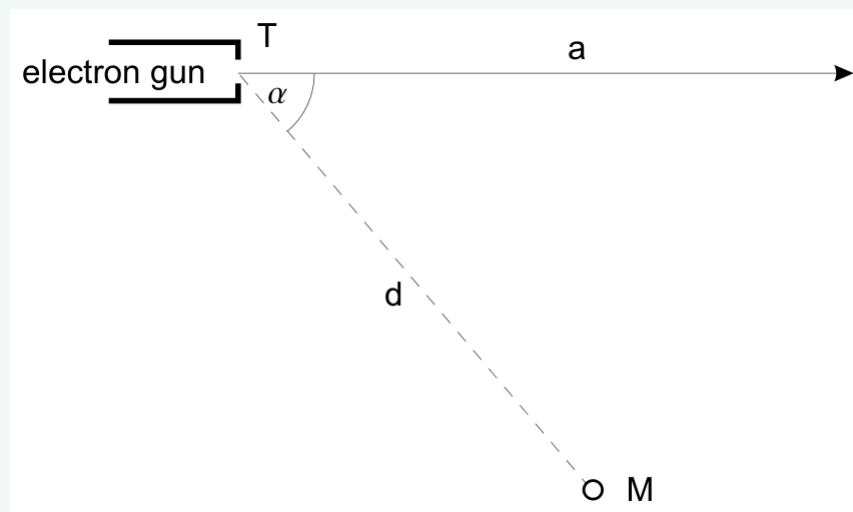


Figure 2:

- a) perpendicular to the plane determined by the line a and the point M
- b) parallel to the segment TM

in order that the electrons hit the target M . Find first the general solution and then substitute the following values: $U = 1000 \text{ V}$, $e = 1.60 \cdot 10^{-19} \text{ C}$, $m_e = 9.11 \cdot 10^{-31} \text{ kg}$, $\alpha = 60^\circ$, $d = 5.0 \text{ cm}$, $B < 0.030 \text{ T}$.

Official English text of the problem.

Answers

- **a)** The field must point **into the page** in Fig. 2, with magnitude

$$B = \frac{2 \sin \alpha}{d} \sqrt{\frac{2m_e U}{e}} \simeq 3.7 \times 10^{-3} \text{ T}.$$

- **b)** Either direction parallel to TM is possible. The allowed magnitudes are

$$B_n = \frac{2\pi n \cos \alpha}{d} \sqrt{\frac{2m_e U}{e}}, \quad n = 1, 2, 3, \dots$$

With the supplied values and $B < 0.030 \text{ T}$, only $n = 1, 2, 3, 4$ are allowed:

$$B \simeq 6.7 \times 10^{-3}, \quad 1.3 \times 10^{-2}, \quad 2.0 \times 10^{-2}, \quad 2.7 \times 10^{-2} \text{ T}.$$

Solution

Let $e > 0$ denote the magnitude of the electron charge, so the charge is $-e$. We use nonrelativistic mechanics, neglect the initial kinetic energy before acceleration, and consider motion in the uniform magnetic field after the electrons leave the gun.

The accelerating potential supplies kinetic energy:

$$\frac{1}{2} m_e v^2 = eU, \quad v = \sqrt{\frac{2eU}{m_e}}.$$

The magnetic force

$$\mathbf{F} = -e \mathbf{v} \times \mathbf{B}$$

is perpendicular to the velocity and therefore does no work. Consequently, the speed remains constant throughout the trajectory.

a) Field perpendicular to the plane of the figure

The velocity remains perpendicular to the magnetic field. The magnetic force provides the centripetal force for circular motion:

$$evB = \frac{m_e v^2}{R}, \quad R = \frac{m_e v}{eB}.$$

Take $T = (0, 0)$, with the initial velocity along the positive x -axis and the positive y -axis upward in the figure. Then

$$M = (d \cos \alpha, -d \sin \alpha).$$

The orbit must bend downward, so its centre is $C = (0, -R)$. Requiring M to lie on this circle gives

$$(d \cos \alpha)^2 + (R - d \sin \alpha)^2 = R^2.$$

Thus

$$d^2 - 2Rd \sin \alpha = 0, \quad R = \frac{d}{2 \sin \alpha}.$$

Combining the geometrical and dynamical expressions for R ,

$$B = \frac{2m_e v \sin \alpha}{ed} = \frac{2 \sin \alpha}{d} \sqrt{\frac{2m_e U}{e}}.$$

To determine the direction, note that an electron initially moving to the right must experience a downward force. Because its charge is negative, the field must point **into the page**.

For the supplied data,

$$v = \sqrt{\frac{2(1.60 \times 10^{-19})(1000)}{9.11 \times 10^{-31}}} \simeq 1.87 \times 10^7 \text{ m s}^{-1},$$

consistent with the nonrelativistic approximation. The required radius is

$$R = \frac{0.050}{2 \sin 60^\circ} \simeq 2.9 \times 10^{-2} \text{ m},$$

and the field is

$$B = 3.6966 \dots \times 10^{-3} \text{ T} \simeq \boxed{3.7 \times 10^{-3} \text{ T}}.$$

This satisfies $B < 0.030 \text{ T}$.

b) Field parallel to TM

Resolve the initial velocity into components along TM and perpendicular to it:

$$v_{\parallel} = v \cos \alpha, \quad v_{\perp} = v \sin \alpha.$$

The parallel component remains constant because the magnetic force has no component along the field. The perpendicular component produces uniform circular motion, so the complete trajectory is a helix whose axis is parallel to TM .

For the circular motion,

$$ev_{\perp} B = \frac{m_e v_{\perp}^2}{r}, \quad \omega = \frac{v_{\perp}}{r} = \frac{eB}{m_e}.$$

Hence its period is

$$P = \frac{2\pi}{\omega} = \frac{2\pi m_e}{eB}.$$

Since T and M differ only in position along the field direction, the electron must return to its initial transverse position when it reaches M . For $v_{\perp} \neq 0$, this happens precisely after an integer number n of complete revolutions:

$$t = nP, \quad n = 1, 2, 3, \dots$$

During this time, the displacement along TM must equal d :

$$d = v \cos \alpha nP = v \cos \alpha \frac{2\pi n m_e}{eB}.$$

Therefore,

$$B_n = \frac{2\pi n m_e v \cos \alpha}{ed} = \frac{2\pi n \cos \alpha}{d} \sqrt{\frac{2m_e U}{e}}, \quad n = 1, 2, 3, \dots$$

Reversing the magnetic field reverses the sense of rotation but leaves both the period and the motion along TM unchanged. Thus **both field directions parallel to TM work**.

For the numerical data, the fundamental field magnitude is

$$B_1 = \frac{2\pi \cos 60^\circ}{0.050} \sqrt{\frac{2(9.11 \times 10^{-31})(1000)}{1.60 \times 10^{-19}}} = 0.0067049 \dots \text{ T}.$$

The upper bound requires

$$nB_1 < 0.030 \text{ T}, \quad n < 4.4743 \dots$$

Thus the allowed integers and corresponding fields, rounded to two significant figures, are

Number of complete revolutions n	Magnetic induction B_n
1	$6.7 \times 10^{-3} \text{ T}$
2	$1.3 \times 10^{-2} \text{ T}$
3	$2.0 \times 10^{-2} \text{ T}$
4	$2.7 \times 10^{-2} \text{ T}$

The next value, $B_5 \simeq 0.0335 \text{ T}$, exceeds the limit.

These discrete solutions apply to the acute, nonzero angle shown. If $\alpha = 0$, the electrons already travel along TM , and any parallel magnetic field leaves that straight motion unchanged. If $\cos \alpha \leq 0$, the electrons cannot reach a target a positive distance along TM , because their velocity component in that direction cannot increase.

The solution was checked against the official solution.