

IPhO 1976

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Problem 1: Small block inside a rotating hollow sphere

A hollow sphere of radius $R = 0.5 \text{ m}$ rotates about a vertical axis through its centre with an angular velocity of $\omega = 5 \text{ s}^{-1}$. Inside the sphere a small block is moving together with the sphere at the height of $R/2$ (Fig. 6). ($g = 10 \text{ m/s}^2$.)

- What should be at least the coefficient of friction to fulfill this condition?
- Find the minimal coefficient of friction also for the case of $\omega = 8 \text{ s}^{-1}$.
- Investigate the problem of stability in both cases,
 - for a small change of the position of the block,
 - for a small change of the angular velocity of the sphere.

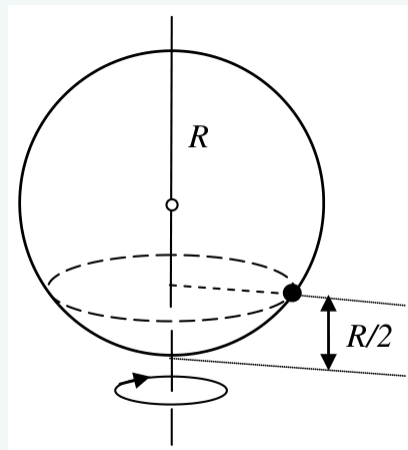


Figure 6

Official English text of the problem.

Answers

- **a)** $\mu_{\min} = \frac{3\sqrt{3}}{23} \approx 0.226$, with friction acting upward along the surface.
- **b)** $\mu_{\min} = \frac{3\sqrt{3}}{29} \approx 0.179$, with friction acting downward along the surface. Both coefficients are dimensionless.

- **c) a)** At the respective minimum coefficients, both equilibria are locally stable, with a restoring tendency on one side and sticking on the other:
- At 5 s^{-1} , an upward displacement produces a downward restoring tendency; a downward displacement permits sticking.
- At 8 s^{-1} , a downward displacement produces an upward restoring tendency; an upward displacement permits sticking.

Azimuthal displacements are neutral. - **c) b)** Comparing nearby steady rotations at the original height: - At 5 s^{-1} , increasing ω permits sticking; decreasing it produces a downward sliding tendency. - At 8 s^{-1} , decreasing ω permits sticking; increasing it produces an upward sliding tendency.

Thus the original height is marginal with respect to changes of speed, while nearby stable resting positions remain available.

Solution

a) Treat the small block as a particle and use Coulomb friction. Let θ be its angular position measured from the downward vertical through the sphere's centre. Its height above the bottom and its distance from the rotation axis are

$$h = R(1 - \cos \theta), \quad \rho = R \sin \theta.$$

The figure specifies $h = R/2$, so

$$\cos \theta_0 = \frac{1}{2}, \quad \theta_0 = 60^\circ, \quad \sin \theta_0 = \frac{\sqrt{3}}{2}.$$

Let N denote the inward normal reaction and let f be the signed friction force, positive **upward along the surface**, in the direction of increasing θ .

In the frame rotating with the sphere, the stationary block experiences gravity, the normal reaction, friction, and the centrifugal force $m\omega^2 R \sin \theta$. Resolving perpendicular and tangential to the surface gives

$$N = m(g \cos \theta + \omega^2 R \sin^2 \theta),$$

$$f = m \sin \theta (g - \omega^2 R \cos \theta).$$

Static equilibrium requires $|f| \leq \mu N$. Therefore, at the specified height,

$$\mu_{\min} = \frac{\sin \theta_0 |g - \omega^2 R \cos \theta_0|}{g \cos \theta_0 + \omega^2 R \sin^2 \theta_0} = \frac{\sqrt{3} |2g - \omega^2 R|}{2g + 3\omega^2 R}.$$

For $\omega = 5 \text{ s}^{-1}$,

$$\omega^2 R = 12.5 \text{ m/s}^2,$$

and hence

$$\mu_{\min} = \frac{\sqrt{3}(20 - 12.5)}{20 + 37.5} = \frac{3\sqrt{3}}{23} \approx 0.226.$$

Here $g - \omega^2 R \cos \theta_0 > 0$, so friction must act upward along the surface.

b) For $\omega = 8 \text{ s}^{-1}$,

$$\omega^2 R = 32 \text{ m/s}^2,$$

giving

$$\mu_{\min} = \frac{\sqrt{3}(32 - 20)}{20 + 96} = \frac{3\sqrt{3}}{29} \approx 0.179.$$

Now $g - \omega^2 R \cos \theta_0 < 0$: the centrifugal force's tangential component exceeds gravity's opposing component. Friction therefore acts downward along the surface.

c) α) Small changes of position

Keep the coefficient equal to the corresponding minimum found above. Consider a small displacement along the surface, followed by release at rest relative to the sphere. We use the idealized single-coefficient Coulomb-friction model.

Introduce

$$q = \frac{\omega^2 R}{g}, \quad K(\theta, q) = \frac{f_{\text{required}}}{N} = \frac{\sin \theta (1 - q \cos \theta)}{\cos \theta + q \sin^2 \theta}.$$

The condition for sticking is

$$-\mu \leq K \leq \mu.$$

If $K > \mu$, even maximum upward friction is insufficient, and the block starts downward. If $K < -\mu$, maximum downward friction is insufficient, and it starts upward.

At fixed angular velocity,

$$\frac{\partial K}{\partial \theta} = \frac{1 - q \cos \theta + q^2 \sin^2 \theta}{(\cos \theta + q \sin^2 \theta)^2}.$$

At $\theta_0 = 60^\circ$, the numerator is

$$1 - \frac{q}{2} + \frac{3q^2}{4} > 0,$$

so K increases with height in both cases.

For 5 s^{-1} , the original equilibrium has $K = \mu$. Raising the block makes $K > \mu$, producing downward motion toward the original latitude. Lowering it places K inside the sticking interval.

For 8 s^{-1} , the original equilibrium has $K = -\mu$. Lowering the block makes $K < -\mu$, producing upward motion toward the original latitude. Raising it places K inside the sticking interval.

Angular velocity	Small upward displacement	Small downward displacement
5 s^{-1}	Downward restoring tendency	Can stick
8 s^{-1}	Can stick	Upward restoring tendency

Thus both equilibria are locally stable under this displacement test. Friction allows neighboring resting positions, so the block need not return to exactly its original latitude. A displacement around the vertical axis changes none of the force components and is neutral.

c) β) Small changes of angular velocity

First compare neighboring steady rotations, keeping the block at its original height. Differentiating the signed friction ratio with respect to q ,

$$\frac{\partial K}{\partial q} = -\frac{\sin \theta}{(\cos \theta + q \sin^2 \theta)^2} < 0.$$

Since $q = \omega^2 R/g$, increasing the angular velocity decreases K .

At 5 s^{-1} , initially $K = \mu$. Increasing ω brings K inside the permitted interval; decreasing ω makes $K > \mu$, requiring more upward friction than is available.

At 8 s^{-1} , initially $K = -\mu$. Decreasing ω brings K inside the permitted interval; increasing ω makes $K < -\mu$, requiring more downward friction than is available.

Initial angular velocity	Small increase in ω	Small decrease in ω
5 s^{-1}	Original height permits sticking	Downward sliding tendency
8 s^{-1}	Upward sliding tendency	Original height permits sticking

The original height therefore cannot be maintained for both signs of a speed change at the minimum coefficient.

Nevertheless, the boundary of the allowed resting positions moves continuously. Keeping $K = \pm\mu$ and differentiating gives

$$\delta\theta_b = \frac{\sin \theta_0}{1 - q \cos \theta_0 + q^2 \sin^2 \theta_0} \delta q, \quad \delta q = \frac{2R\omega}{g} \delta\omega.$$

Its coefficient is finite and positive in both cases. Thus, in the unfavorable direction of speed change, a small downward adjustment in case a), or a small upward adjustment in case b), reaches the neighboring range of admissible resting positions.

The solution was checked against the official solution.

Problem 2: Helium in a cylinder with a piston and a valved dividing wall

The walls of a cylinder of base 1 dm^2 , the piston and the inner dividing wall are perfect heat insulators (Fig. 10). The valve in the dividing wall opens if the pressure on the right side is greater than on the left side. Initially there is 12 g helium in the left side and 2 g helium in the right side. The lengths of both sides are 11.2 dm each and the temperature is 0°C . Outside we have a pressure of 100 kPa .

The specific heat at constant volume is $c_v = 3.15 \text{ J/gK}$, at constant pressure it is $c_p = 5.25 \text{ J/gK}$. The piston is pushed slowly towards the dividing wall. When the valve opens we stop then continue pushing slowly until the wall is reached. Find the work done on the piston by us.

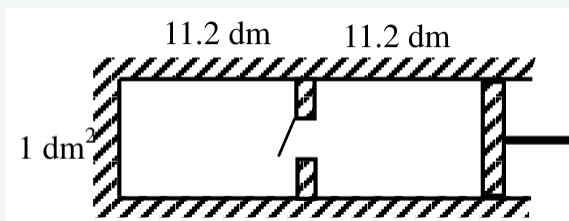


Figure 10

Answers

- **Problem 2:** The work done on the piston by us is

$$W_{\text{us}} \approx 3.68 \times 10^3 \text{ J} = 3.68 \text{ kJ}.$$

Solution

Treat helium as an ideal gas with the given constant specific heats. Neglect friction and kinetic energy; the slow piston motion makes the compression quasistatic.

The initial volume of each compartment is

$$V_0 = A\ell_0 = (1 \text{ dm}^2)(11.2 \text{ dm}) = 1.12 \times 10^{-2} \text{ m}^3.$$

Also,

$$T_0 = 273.15 \text{ K}, \quad R_s = c_p - c_v = 2.10 \text{ J g}^{-1} \text{ K}^{-1}, \quad \gamma = \frac{c_p}{c_v} = \frac{5}{3}.$$

Before the valve opens. The left-hand gas remains at fixed volume and temperature, so its pressure stays at

$$p_{L0} = \frac{m_L R_s T_0}{V_0} = 614.6 \text{ kPa}.$$

The initial right-hand pressure is

$$p_{R0} = \frac{m_R R_s T_0}{V_0} = 102.4 \text{ kPa}, \quad \frac{p_{L0}}{p_{R0}} = \frac{12}{2} = 6.$$

The right-hand gas undergoes quasistatic adiabatic compression:

$$p_R v^\gamma = p_{R0} V_0^\gamma,$$

where v is its instantaneous volume. In the quasistatic limit, the valve starts opening when $p_R = p_{L0}$. Hence the right-hand volume at opening is

$$v_* = V_0 \left(\frac{p_{R0}}{p_{L0}} \right)^{1/\gamma} = V_0 6^{-3/5} = 3.822 \times 10^{-3} \text{ m}^3.$$

The pause contributes no work because the piston does not move. Any internal equilibration during the pause also leaves the total energy unchanged.

After the valve opens. Continued slow compression transfers helium into the left compartment. The pressure difference needed to open the ideal valve is infinitesimal, so the two compartments have a common pressure p .

Their temperatures need not be equal. For either compartment, the ideal-gas equation gives

$$U_i = m_i c_v T_i = \frac{c_v}{R_s} p V_i = \frac{p V_i}{\gamma - 1}.$$

Thus, writing the total gas volume as $\mathcal{V} = V_0 + v$,

$$U = \frac{p\mathcal{V}}{\gamma - 1}.$$

The whole gas system is thermally insulated, and only the piston does boundary work. Therefore,

$$dU = -p d\mathcal{V}.$$

Substitution gives

$$\frac{\mathcal{V} dp + p d\mathcal{V}}{\gamma - 1} = -p d\mathcal{V},$$

or

$$\mathcal{V} dp + \gamma p d\mathcal{V} = 0.$$

Consequently,

$$p\mathcal{V}^\gamma = \text{constant}.$$

This relation follows from energy conservation and equal pressures; it does not require equal compartment temperatures.

At the start of this stage, $\mathcal{V} = V_0 + v_*$ and $p = p_{L0}$. When the piston reaches the dividing wall, all the helium occupies V_0 . Its final pressure is therefore

$$\begin{aligned} p_f &= p_{L0} \left(\frac{V_0 + v_*}{V_0} \right)^\gamma \\ &= p_{L0} \left(1 + 6^{-3/5} \right)^{5/3} \\ &\approx 1.00257 \times 10^6 \text{ Pa}. \end{aligned}$$

Work supplied by us. Since no heat enters the gas, the total work done on it equals its internal-energy increase:

$$\begin{aligned} W_{\text{on gas}} &= U_f - U_i \\ &= \frac{p_f V_0 - (p_{L0} + p_{R0}) V_0}{\gamma - 1} \\ &\approx 4797.3 \text{ J}. \end{aligned}$$

The atmosphere supplies part of this work. The piston sweeps out the entire initial right-hand volume, so

$$W_{\text{atmosphere}} = p_{\text{ext}} V_0 = (100000)(0.0112) = 1120 \text{ J}.$$

With negligible piston kinetic energy and friction,

$$W_{\text{us}} + W_{\text{atmosphere}} = W_{\text{on gas}}.$$

Hence

$$W_{\text{us}} = 4797.3 - 1120 \approx 3.68 \times 10^3 \text{ J}.$$

The solution was checked against the official solution.

Problem 3: Measuring an air bubble inside a glass sphere

Somewhere in a glass sphere there is an air bubble. Describe methods how to determine the diameter of the bubble without damaging the sphere.

Official English text of the problem.

Answers

Problem 3. Three possible nondestructive methods are:

- **Weighing:** Measure the sphere's outer diameter D and mass m . If the glass density is ρ_g , neglecting the mass of air,

$$d = \left(D^3 - \frac{6m}{\pi\rho_g} \right)^{1/3}.$$

- **Optical immersion:** Immerse the sphere in a liquid with the same refractive index as the glass. Measure the bubble's transverse outline with calibrated optics. If its image diameter is L and the transverse magnification is M ,

$$d = \frac{L}{M}.$$

- **Radiography:** Use a parallel X-ray beam and measure the bubble's projected diameter. With a calibrated image scale, this gives its actual diameter directly.

No numerical data are supplied.

Solution

We assume that the glass is homogeneous and that the bubble is spherical, as is implicit in asking for its diameter. Write the outer radius as $R = D/2$ and the bubble radius as $r = d/2$. The bubble need not be at the centre of the glass sphere.

Weighing

Measure D with calipers and m with a balance. The glass density must be known independently, for example from a separate specimen of the same glass.

The outer volume and bubble volume are

$$V = \frac{\pi D^3}{6}, \quad v = \frac{\pi d^3}{6}.$$

If the density of the air inside the bubble is ρ_a , conservation of mass gives

$$m = \rho_g(V - v) + \rho_a v = \rho_g V - (\rho_g - \rho_a)v.$$

Consequently,

$$d = \left[\frac{\rho_g D^3 - 6m/\pi}{\rho_g - \rho_a} \right]^{1/3}.$$

Ordinarily $\rho_a \ll \rho_g$, giving

$$d = \left(D^3 - \frac{6m}{\pi\rho_g} \right)^{1/3}.$$

The result is independent of the bubble's position. For a small bubble, this method requires precise measurements: the missing mass is only approximately the fraction $(d/D)^3$ of the mass of a solid glass sphere.

Optical measurement in an index-matching liquid

Place the sphere in a transparent tank containing a liquid whose refractive index equals that of the glass. Use monochromatic light if necessary so that dispersion does not prevent accurate matching.

At the outer surface, Snell's law becomes

$$n_g \sin i = n_\ell \sin t.$$

Since $n_\ell = n_g$, we have $t = i$: the outer spherical boundary no longer bends the rays. The bubble remains visible because the air and glass have different refractive indices.

View through a flat tank window, perpendicular to the viewing direction. Use parallel viewing rays—for example, a telecentric imaging system—or a travelling microscope whose axis remains perpendicular to the window.

The limiting rays defining the bubble's outline are tangent to the bubble. Approached from outside the outline, these rays remain entirely in the glass and matching liquid, so they are not refracted by the air cavity. The perpendicular projection of a sphere is a circle of its true diameter, regardless of its depth.

Thus, if a travelling microscope records the two opposite edges at transverse positions x_L and x_R ,

$$d = |x_R - x_L|.$$

Alternatively, a calibrated image with transverse magnification M gives

$$d = L/M.$$

X-ray transmission

Illuminate the sphere with a sufficiently penetrating, approximately parallel X-ray beam and record its transmitted intensity. Neglect ray bending and scattering. Air attenuates the beam much less than glass, so the bubble produces a region of enhanced transmission.

For a ray at transverse distance s from the projected bubble centre, the length of glass replaced by air is

$$\ell_b(s) = \begin{cases} 2\sqrt{r^2 - s^2}, & s < r, \\ 0, & s \geq r. \end{cases}$$

If μ_g and μ_a are the linear attenuation coefficients, the Beer–Lambert law gives

$$\frac{I}{I_{\text{solid}}} = \exp[(\mu_g - \mu_a)\ell_b(s)],$$

where I_{solid} is the transmission through an otherwise identical solid glass sphere.

The enhanced-transmission region therefore ends at $s = r$, giving a circular projection of diameter d . With parallel rays, the bubble's position along the beam does not affect its projected size. If the recorded outer-sphere and bubble diameters are L_s and L_b , respectively,

$$d = D \frac{L_b}{L_s}.$$

The solution was checked against the official solution.