

IPhO 1975

Theory problems and solutions · Güstrow, German Democratic Republic · 3 theory problems with complete solutions

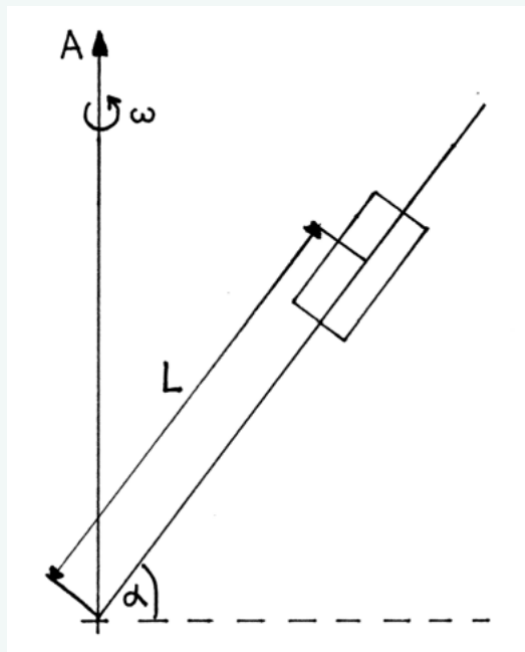
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Problem 1: Rotating rod

A rod revolves with a constant angular velocity ω around a vertical axis A. The rod includes a fixed angle of $\pi/2 - \alpha$ with the axis. A body of mass m can glide along the rod. The coefficient of friction is $\mu = \tan \beta$. The angle β is called "friction angle".

a) Determine the angles α under which the body remains at rest and under which the body is in motion if the rod is not rotating (i.e. $\omega = 0$).

b) The rod rotates with constant angular velocity $\omega > 0$. The angle α does not change during rotation. Find the condition for the body to remain at rest relative to the rod.



You can use the following relations:

$$\sin(\alpha \pm \beta) = \sin \alpha \cdot \cos \beta \pm \cos \alpha \cdot \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cdot \cos \beta \mp \sin \alpha \cdot \sin \beta$$

Official English text of the problem.

Answers

Take $0 \leq \alpha \leq \pi/2$ and $0 \leq \beta < \pi/2$. Let $L \geq 0$ denote the distance along the rod from its intersection with the rotation axis to the body.

- **a)** A body initially at rest remains at rest when

$$0 \leq \alpha \leq \beta.$$

It slides down the rod when

$$\beta < \alpha \leq \frac{\pi}{2}.$$

The case $\alpha = \beta$ is limiting equilibrium.

- **b)** The necessary and sufficient condition for rest relative to the rod is

$$|g \sin \alpha - \omega^2 L \cos^2 \alpha| \leq \tan \beta \cos \alpha (g + \omega^2 L \sin \alpha).$$

For $0 \leq \alpha < \pi/2$, this is equivalently $L_{\min} \leq L \leq L_{\max}$, where

$$L_{\min} = \frac{g}{\omega^2 \cos \alpha} \max\{0, \tan(\alpha - \beta)\},$$

$$L_{\max} = \begin{cases} \frac{g \tan(\alpha + \beta)}{\omega^2 \cos \alpha}, & \alpha + \beta < \frac{\pi}{2}, \\ \infty, & \alpha + \beta \geq \frac{\pi}{2}. \end{cases}$$

For a vertical rod, $\alpha = \pi/2$, rest is impossible.

Solution

Model the body as a particle constrained to a rigid rod in uniform gravity. Use Coulomb static friction: the friction force can take either direction along the rod, with magnitude at most μN , where N is the normal reaction. We determine whether a body initially at rest relative to the rod can remain so.

a) Nonrotating rod

The rod makes an angle α with the horizontal. Gravity therefore has components

$$mg \sin \alpha \quad \text{down the rod,} \quad mg \cos \alpha \quad \text{perpendicular to the rod.}$$

There is no acceleration perpendicular to the rod, so

$$N = mg \cos \alpha.$$

For equilibrium, friction must balance the component of gravity along the rod:

$$mg \sin \alpha \leq \mu mg \cos \alpha.$$

For $0 \leq \alpha < \pi/2$, this becomes

$$\tan \alpha \leq \mu = \tan \beta,$$

and hence $\alpha \leq \beta$. At equality, static friction reaches its maximum value.

If $\alpha > \beta$, friction cannot balance gravity, and the body released from rest slides down the rod. At $\alpha = \pi/2$, the normal reaction and thus the available friction vanish, so the body also falls.

b) Rotating rod

Use a frame rotating with the rod. A body at rest in this frame experiences no Coriolis force; there is also no Euler force because ω is constant. In addition to gravity and the contact forces, include the centrifugal force.

The body's distance from the vertical axis is

$$r = L \cos \alpha,$$

so the centrifugal force is horizontally outward, with magnitude

$$F_c = m\omega^2 r = m\omega^2 L \cos \alpha.$$

Take the positive direction along the rod to be outward and upward. The combined component of gravity and centrifugal force along the rod is

$$F_{\parallel} = F_c \cos \alpha - mg \sin \alpha = m (\omega^2 L \cos^2 \alpha - g \sin \alpha).$$

Their components perpendicular to the rod act in the same direction. Consequently, the normal reaction required for equilibrium is

$$N = mg \cos \alpha + F_c \sin \alpha = m \cos \alpha (g + \omega^2 L \sin \alpha).$$

Friction must balance $F_{\parallel}$, so equilibrium is possible precisely when

$$|F_{\parallel}| \leq \mu N.$$

Substituting and cancelling m gives

$$|g \sin \alpha - \omega^2 L \cos^2 \alpha| \leq \tan \beta \cos \alpha (g + \omega^2 L \sin \alpha).$$

Thus rotation changes both the force that friction must balance and the normal reaction that determines the available friction.

To express the condition as bounds on L , split the absolute-value inequality into its two inequalities. Multiply by $\cos \beta > 0$, and use the supplied addition formulas. This gives

$$\begin{aligned}\omega^2 L \cos \alpha \cos(\alpha - \beta) &\geq g \sin(\alpha - \beta), \\ \omega^2 L \cos \alpha \cos(\alpha + \beta) &\leq g \sin(\alpha + \beta).\end{aligned}$$

For $0 \leq \alpha < \pi/2$, both $\cos \alpha$ and $\cos(\alpha - \beta)$ are positive. The first inequality therefore requires

$$L \geq \frac{g \tan(\alpha - \beta)}{\omega^2 \cos \alpha}.$$

When $\alpha \leq \beta$, the right-hand side is nonpositive, so this imposes no restriction beyond $L \geq 0$. Combining these cases yields the stated $L_{\min}$.

For the second inequality:

- If $\alpha + \beta < \pi/2$, its coefficient of L is positive, giving

$$L \leq \frac{g \tan(\alpha + \beta)}{\omega^2 \cos \alpha}.$$

- If $\alpha + \beta \geq \pi/2$, its left-hand side is nonpositive while its right-hand side is positive. It is therefore satisfied for every $L \geq 0$, and there is no upper bound.

Below a positive $L_{\min}$, gravity cannot be balanced and the body tends to slide inward and downward. Above a finite $L_{\max}$, the outward force cannot be balanced and the body tends to slide outward and upward. Equality at either bound is included as limiting equilibrium.

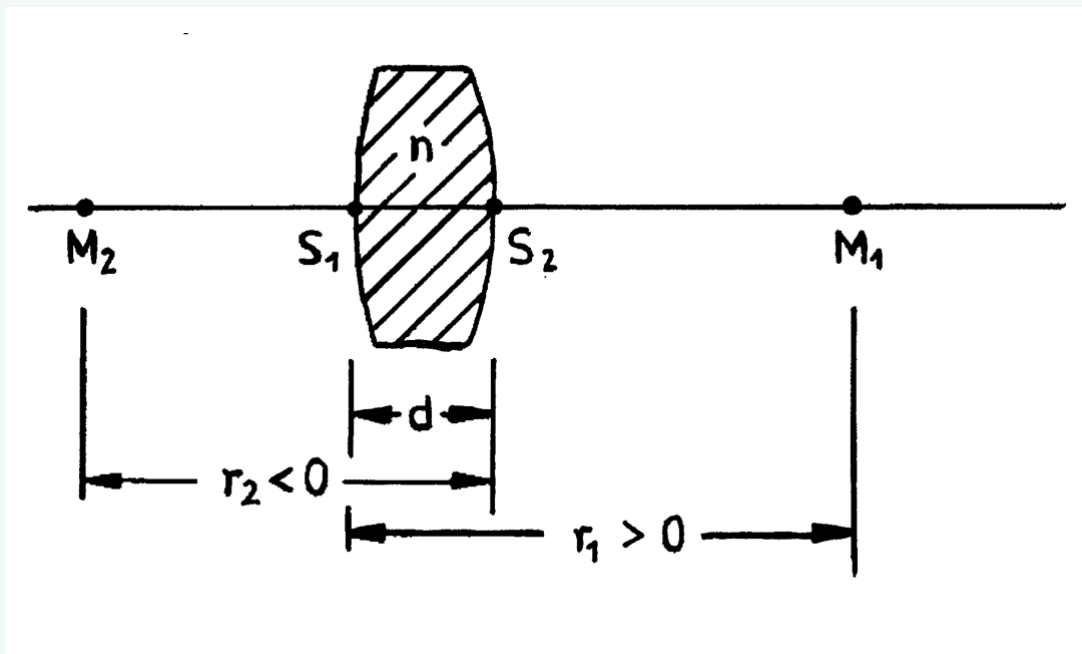
Finally, for $\alpha = \pi/2$, the body lies on the rotation axis. The centrifugal force and normal reaction vanish, leaving gravity unbalanced.

The solution was checked against the official solution.

Problem 2: Thick lens

The focal length f of a thick glass lens in air with refractive index n , radius curvatures r_1, r_2 and vertex distance d (see figure) is given by:

$$f = \frac{nr_1r_2}{(n-1)[n(r_2-r_1)+d(n-1)]}$$



Remark: $r_i > 0$ means that the central curvature point M_i is on the right side of the aerial vertex S_i , $r_i < 0$ means that the central curvature point M_i is on the left side of the aerial vertex S_i ($i = 1, 2$).

For some special applications it is required, that the focal length is independent from the wavelength.

- For how many different wavelengths can the same focal length be achieved?
- Describe a relation between r_i ($i = 1, 2$), d and the refractive index n for which the required wavelength independence can be fulfilled and discuss this relation.

Sketch possible shapes of lenses and mark the central curvature points M_1 and M_2 .

- Prove that for a given planconvex lens a specific focal length can be achieved by only one wavelength.

d) State possible parameters of the thick lens for two further cases in which a certain focal length can be realized for one wavelength only. Take into account the physical and the geometrical circumstances.

Official English text of the problem.

Answers

- **a)** At most **two distinct wavelengths**, assuming that distinct wavelengths have distinct refractive indices in the spectral interval considered.
- **b)** For two distinct indices $n_1, n_2 > 1$, exact equality of focal lengths requires

$$r_1 - r_2 = d \left(1 - \frac{1}{n_1 n_2} \right).$$

First-order wavelength independence at an index n_0 requires

$$r_1 - r_2 = d \left(1 - \frac{1}{n_0^2} \right).$$

Consequently, $0 < r_1 - r_2 < d$. Possible forms are an unusually thick biconvex lens and two mutually reversed meniscus lenses. Their curvature centres satisfy M_1 to the left of M_2 .

- **c)** For a planconvex lens with convex-surface radius $R > 0$,

$$f = \frac{R}{n - 1}.$$

Every attainable focal length therefore determines exactly one refractive index and hence one wavelength.

- **d)** Two physically realizable examples, with $R > 0$ an arbitrary length, are

$$r_1 = R, \quad r_2 = -R, \quad d = R$$

and

$$r_1 = -R, \quad r_2 = R, \quad d = R.$$

These are biconvex and biconcave lenses, respectively. For each attainable focal length, the other algebraic index is negative in the first example and less than 1 in the second.

Solution

We use the supplied formula within **paraxial geometrical optics**, with fixed lens geometry and homogeneous glass. Wavelength enters through $n = n(\lambda)$. For ordinary glass in a transparent interval of normal dispersion, $n > 1$ and $n(\lambda)$ is one-to-one.

Define the optical power and two convenient constants:

$$\Phi(n) = \frac{1}{f(n)}, \quad P = r_1 r_2, \quad C = d + r_2 - r_1.$$

For finite, nonzero radii, the given formula becomes

$$\Phi(n) = \frac{1}{P} \left[Cn - (C + d) + \frac{d}{n} \right]. \quad (1)$$

a) Number of wavelengths

For a prescribed finite focal length F , equation (1) gives

$$Cn^2 - \left(C + d + \frac{P}{F} \right) n + d = 0. \quad (2)$$

This is quadratic in n , so there can be at most two distinct refractive indices. If $C = 0$, it becomes linear.

Both roots can be physically admissible: part (b) gives a construction for any chosen pair of distinct glass indices. Thus the maximum is **two wavelengths**.

Exact constancy throughout a continuous wavelength interval is impossible for a nondegenerate dispersive lens: equation (2) cannot hold identically in n , since its constant term is $d > 0$.

b) Conditions and possible lens shapes

For two distinct indices n_1 and n_2 , equality of focal lengths is equivalent to equality of optical powers. Subtracting equation (1) at the two indices,

$$P[\Phi(n_1) - \Phi(n_2)] = (n_1 - n_2) \left(C - \frac{d}{n_1 n_2} \right).$$

Since $n_1 \neq n_2$,

$$C = \frac{d}{n_1 n_2}.$$

Therefore,

$$r_1 - r_2 = d \left(1 - \frac{1}{n_1 n_2} \right), \quad d = \frac{n_1 n_2}{n_1 n_2 - 1} (r_1 - r_2). \quad (3)$$

Because $n_1, n_2 > 1$,

$$0 < r_1 - r_2 < d. \quad (4)$$

For example, given any two available glass indices, choose $r_1 = R$, $r_2 = -R$, and

$$d = \frac{2Rn_1 n_2}{n_1 n_2 - 1}.$$

This gives a biconvex lens satisfying equation (3), establishing that two physical solutions are possible.

For a fixed lens, the partner index is

$$n_2 = \frac{d}{Cn_1}.$$

Both indices must lie within the actual dispersion range of the glass.

To obtain **first-order independence near one wavelength**, require

$$\frac{df}{d\lambda} = \frac{df}{dn} \frac{dn}{d\lambda} = 0.$$

Since ordinary dispersion has $dn/d\lambda \neq 0$, this requires $df/dn = 0$, or equivalently $d\Phi/dn = 0$ for finite f . Equation (1) gives

$$\frac{d\Phi}{dn} = \frac{1}{P} \left(C - \frac{d}{n^2} \right).$$

Thus, at the design index n_0 ,

$$\boxed{C = \frac{d}{n_0^2}, \quad r_1 - r_2 = d \left(1 - \frac{1}{n_0^2} \right).} \quad (5)$$

The exact two-wavelength condition and the stationary condition are related by

$$n_0 = \sqrt{n_1 n_2}.$$

The two distinct indices therefore lie on opposite sides of the stationary index. At the stationary focal length itself, the quadratic has a double root: there is one exact wavelength, with vanishing first-order variation.

The remaining dependence is generally second order, because

$$\frac{d^2\Phi}{dn^2} = \frac{2d}{Pn^3} \neq 0.$$

Substitution into the focal-length formula gives

$$\boxed{f(n_0) = -\frac{n_0^2 r_1 r_2}{d(n_0 - 1)^2}.} \quad (6)$$

Thus the admissible biconvex form has positive focal length, while the admissible meniscus forms have negative focal length.

For precise descriptions of the sketches, put

$$x(S_1) = 0, \quad x(S_2) = d.$$

The curvature centres are then

$$x(M_1) = r_1, \quad x(M_2) = d + r_2.$$

Under equation (5),

$$\boxed{x(M_2) - x(M_1) = \frac{d}{n_0^2} > 0.} \quad (7)$$

Let $\delta = d(1 - 1/n_0^2)$, so $0 < \delta < d$. The following choices give three explicit sketches; the listed points are ordered from left to right.

Lens shape	Possible radii	Order of vertices and centres
Biconvex: both optical faces bulge outward	$r_1 = \delta/2, \quad r_2 = -\delta/2$	S_1, M_1, M_2, S_2
Meniscus: first face convex, second face concave	$r_1 = d + \delta, \quad r_2 = d$	S_1, S_2, M_1, M_2

Lens shape	Possible radii	Order of vertices and centres
Reversed meniscus: first face concave, second face convex	$r_1 = -d, \quad r_2 = -(d + \delta)$	M_1, M_2, S_1, S_2

The optical faces may be small spherical caps joined by a side wall. In the biconvex case, condition (4) becomes

$$d > r_1 + |r_2|.$$

It therefore requires a long piece of glass with convex end faces, rather than the usual thin biconvex shape. A biconcave lens has $r_1 < 0 < r_2$, contradicting $r_1 > r_2$, so it cannot satisfy the achromatic condition.

c) Planconvex lens

There are two orientations. Taking the plane-surface limit gives

$$r_2 \rightarrow \infty : \quad f = \frac{r_1}{n-1},$$

or

$$r_1 \rightarrow \infty : \quad f = -\frac{r_2}{n-1}.$$

For a convex spherical surface of radius magnitude R , both reduce to

$$f(n) = \frac{R}{n-1}.$$

The thickness drops out of the effective focal length. Moreover,

$$\frac{df}{dn} = -\frac{R}{(n-1)^2} < 0 \quad (n > 1).$$

Hence $f(n)$ is strictly monotonic, and a specified attainable focal length determines

$$\boxed{n = 1 + \frac{R}{F}}.$$

There is therefore exactly one corresponding wavelength within a one-to-one dispersion interval.

d) Two further cases with only one physical wavelength

For $C \neq 0$, equation (2) shows that the product of the two algebraic indices is

$$n \tilde{n} = \frac{d}{C}. \quad (8)$$

This identifies two useful families.

1. $r_1 - r_2 \geq d$, or $C \leq 0$.

If $C < 0$, a physical root $n > 1$ has a negative partner,

$$\tilde{n} = \frac{d}{Cn} < 0,$$

which cannot describe ordinary glass. If $C = 0$, equation (2) is linear and again has at most one solution.

A concrete example is

$$r_1 = R, \quad r_2 = -R, \quad d = R.$$

This is a physically realizable biconvex lens. Its focal length is

$$f(n) = \frac{Rn}{n^2 - 1},$$

and

$$\frac{df}{dn} = -\frac{R(n^2 + 1)}{(n^2 - 1)^2} < 0.$$

Every attainable positive focal length therefore occurs at one physical index only. Its other algebraic root is $\tilde{n} = -1/n$.

2. $r_1 \leq r_2$, or $C \geq d$.

For a physical root $n > 1$,

$$0 < \tilde{n} = \frac{d}{Cn} \leq \frac{1}{n} < 1.$$

The second root is therefore outside the ordinary-glass range.

A concrete example is

$$r_1 = -R, \quad r_2 = R, \quad d = R.$$

This is a biconcave lens with positive central thickness; its thickness increases away from the axis. Its focal length is

$$f(n) = -\frac{Rn}{(n-1)(3n-1)} < 0,$$

and its other algebraic index is

$$\tilde{n} = \frac{1}{3n} < 1.$$

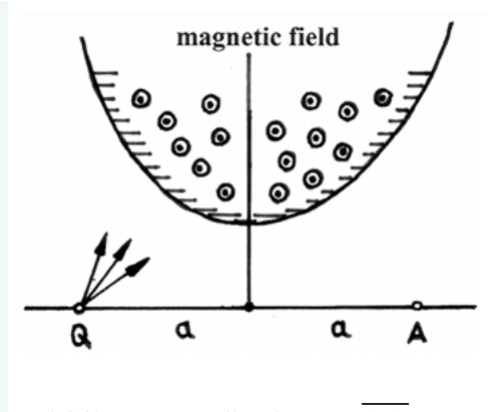
Thus every attainable negative focal length also corresponds to only one physical wavelength.

Both examples have finite thickness and two curved surfaces. A sufficiently small aperture makes them consistent with the paraxial approximation.

The solution was checked against the official solution.

Problem 3: Ions in a magnetic field

A beam of positive ions (charge $+e$) of the same and constant mass m spread from point Q in different directions in the plane of paper (see figure²). The ions were accelerated by a voltage U . They are deflected in a uniform magnetic field B that is perpendicular to the plane of paper. The boundaries of the magnetic field are made in a way that the initially diverging ions are focused in point A ($\overline{QA} = 2 \cdot a$). The trajectories of the ions are symmetric to the middle perpendicular on $\overline{QA}$.



² Remark: This illustrative figure was not part of the original problem formulation.

Among different possible boundaries of magnetic fields a specific type shall be considered in which a contiguous magnetic field acts around the middle perpendicular and in which the points Q and A are in the field free area.

- Describe the radius curvature R of the particle path in the magnetic field as a function of the voltage U and the induction B .
- Describe the characteristic properties of the particle paths in the setup mentioned above.
- Obtain the boundaries of the magnetic field boundaries by geometrical constructions for the cases $R < a$, $R = a$ and $R > 0$.
- Describe the general equation for the boundaries of the magnetic field.

Official English text of the problem.

Answers

Take the midpoint of QA as the origin, with $Q = (-a, 0)$, $A = (a, 0)$, and the illustrated beam above the x -axis.

- a)

$$R = \frac{mv}{eB} = \frac{1}{B} \sqrt{\frac{2mU}{e}}.$$

- b) Each trajectory consists of a straight incoming segment, a circular arc of radius R , and a straight outgoing segment. The circle's centre lies on the perpendicular bisector of QA . If the initial direction makes an angle θ with QA , the magnetic deflection is 2θ , and each straight segment has length

$$\ell = \frac{a - R \sin \theta}{\cos \theta}.$$

- c) Construct the circle centre by intersecting the perpendicular bisector with a line parallel to the incoming ray, displaced by R towards the magnetic-force side. Its perpendicular foot on the ray is the entry point; reflection gives the exit point.

For the forward-going rays:

Case	Allowed angles with $\ell > 0$
$R < a$	$0 < \theta < \pi/2$
$R = a$	$0 < \theta < \pi/2$
$R > a$	$0 < \theta < \arcsin(a/R)$

For $R < a$, the complete boundary branches have vertical asymptotes $x = \pm R$. For $R \geq a$, they have limiting endpoints at Q, A ; a finite angular aperture keeps those points strictly outside the field.

- **d)** The portions of the boundary traversed by the beam are

$$x = \pm R \sin \theta, \quad y = (a - R \sin \theta) \tan \theta$$

or, equivalently,

$$y = \frac{|x|(a - |x|)}{\sqrt{R^2 - x^2}}, \quad 0 \leq |x| < \min(a, R).$$

Their implicit equation is

$$(R^2 - x^2)y^2 = x^2(a - |x|)^2,$$

with the branch restrictions above.

Solution

a) Radius of curvature

Use the nonrelativistic approximation, neglect the ions' initial kinetic energy before acceleration, and neglect interactions between ions. Acceleration through the potential difference U gives

$$\frac{1}{2}mv^2 = eU, \quad v = \sqrt{\frac{2eU}{m}}.$$

The magnetic force is perpendicular to the velocity, so it changes the direction of motion without changing the speed. Inside the uniform field it supplies the centripetal force:

$$evB = \frac{mv^2}{R}.$$

Consequently,

$$R = \frac{mv}{eB} = \sqrt{\frac{2mU}{eB^2}}.$$

Here B denotes the field magnitude. With SI inputs, R is in metres.

b) Geometry of the trajectories

Treat the field boundary as sharp, neglecting fringe fields. Let O be the midpoint of QA , and choose

$$Q = (-a, 0), \quad A = (a, 0).$$

For the outward field direction illustrated, positive ions travelling upwards bend clockwise. Consider an initial direction

$$\mathbf{v} = v(\cos \theta, \sin \theta), \quad 0 < \theta < \frac{\pi}{2}.$$

Let P be the entry point, P' the exit point, and C the centre of the circular arc.

Outside the field, the trajectory is straight. At a sharp boundary, the velocity remains continuous, so the straight and circular portions join tangentially. Symmetry about the y -axis requires P' to be the reflection of P , and the circle centre must lie on that axis.

Writing $QP = \ell$,

$$P = (-a + \ell \cos \theta, \ell \sin \theta).$$

The radius from P towards C points in the magnetic-force direction:

$$C - P = R(\sin \theta, -\cos \theta).$$

Since $C_x = 0$,

$$-a + \ell \cos \theta + R \sin \theta = 0,$$

and therefore

$$\ell = \frac{a - R \sin \theta}{\cos \theta}.$$

It follows that

$$\begin{aligned} P &= (-R \sin \theta, (a - R \sin \theta) \tan \theta), \\ P' &= (R \sin \theta, (a - R \sin \theta) \tan \theta). \end{aligned}$$

The centre is

$$C = \left(0, \frac{a \sin \theta - R}{\cos \theta} \right).$$

The outgoing velocity has direction $(\cos \theta, -\sin \theta)$. A straight segment of length ℓ then carries the ion from P' to A , because

$$P' + \ell(\cos \theta, -\sin \theta) = (a, 0).$$

The velocity thus turns through 2θ inside the field. The circular arc subtends the same angle at C , and its chord has length

$$PP' = 2R \sin \theta.$$

c) Geometrical construction and the three regimes

For a chosen incoming direction, the construction is:

1. Draw the ray from Q .
2. Draw a parallel line a perpendicular distance R away, on the side towards which the magnetic force bends the ion.
3. Its intersection with the perpendicular bisector of QA is C .
4. Drop the perpendicular from C to the incoming ray. Its foot is P .
5. Reflect P in the perpendicular bisector to obtain P' , and draw the clockwise circular arc of radius R joining them.

Varying the incoming direction traces the entrance boundary. Reflection traces the exit boundary. The perpendicular foot must lie ahead of Q , so

$$\ell > 0 \iff R \sin \theta < a.$$

For describing the shapes, put $s = |x|$.

- $R < a$.

Every acute incoming angle is admissible. As θ increases from 0 towards $\pi/2$, the boundary runs from O towards infinite height, approaching

$$\boxed{x = -R \quad \text{and} \quad x = R}$$

on the two sides. Since $a > R$, Q and A lie outside this strip.

Near O ,

$$y \sim \frac{a}{R}|x|,$$

so the complete limiting locus has a pointed bottom.

The branches need not rise monotonically throughout. If

$$\frac{R}{a} > \sqrt{\frac{27}{32}},$$

each branch has a local maximum followed by a local minimum before rising towards its asymptote. Otherwise it rises monotonically. These features follow from the derivative given in part (d).

- $R = a$.

The construction works for every acute angle, and

$$y = s \sqrt{\frac{a-s}{a+s}}, \quad 0 \leq s < a.$$

Each branch starts at O , rises to a maximum, and returns towards the corresponding source or image point as $\theta \rightarrow \pi/2$.

The maximum occurs at

$$s = \frac{\sqrt{5}-1}{2} a,$$

with height

$$y_{\max} = a \left(\frac{\sqrt{5}-1}{2} \right)^{5/2} \approx 0.3003a.$$

The limiting circular trajectory is the upper semicircle of radius a centred at O , with endpoints Q , A .

- $R > a$ — the likely intended third case.

The construction requires

$$0 < \theta < \alpha, \quad \boxed{\alpha = \arcsin(a/R)}.$$

Geometrically, α is obtained from a right triangle with hypotenuse R and opposite side a .

Each boundary branch starts at O , rises to one maximum, and returns towards Q or A as $\theta \rightarrow \alpha$. At that limit, $\ell \rightarrow 0$, and the circular trajectory has centre

$$C = (0, -\sqrt{R^2 - a^2}).$$

Its highest point is

$$y = R - \sqrt{R^2 - a^2}.$$

For $R \geq a$, using the complete limiting locus would put Q, A on the field boundary. To keep them strictly within field-free areas, choose a beam whose maximum angle is strictly below the limiting value. Continue the unused boundary around its trajectories; for example, close the field above them with a circular arc constructed for a slightly larger admissible angle. The resulting field is connected and remains separated from Q, A .

Only boundary portions actually crossed by the beam are fixed by the focusing condition.

d) General boundary equation

From part (b),

$$s = R \sin \theta, \quad y = (a - s) \tan \theta.$$

For the acute-angle branch,

$$\tan \theta = \frac{s}{\sqrt{R^2 - s^2}}.$$

Eliminating θ gives

$$y = f(s) = \frac{s(a - s)}{\sqrt{R^2 - s^2}}, \quad s = |x|, \quad 0 \leq s < \min(a, R).$$

Hence

$$(R^2 - x^2)y^2 = x^2(a - |x|)^2.$$

The unsquared expression and domain select the physical upper boundary. Squaring alone also admits other branches.

For completeness, differentiating gives

$$f'(s) = \frac{s^3 - 2R^2s + aR^2}{(R^2 - s^2)^{3/2}}.$$

Thus all stationary heights mentioned in part (c) satisfy

$$s^3 - 2R^2s + aR^2 = 0.$$

For $R > a$, this equation has exactly one root in $0 < s < a$, giving the maximum of each branch.

We can also verify that the constructed boundary keeps each trajectory in the appropriate region. Write the centre height for entry coordinate s as

$$c(s) = f(s) - \sqrt{R^2 - s^2} = \frac{as - R^2}{\sqrt{R^2 - s^2}}.$$

Then

$$c'(s) = \frac{R^2(a - s)}{(R^2 - s^2)^{3/2}} > 0.$$

An ion entering at $s = s_0$ follows the upper circular arc

$$y_{\text{arc}}(s) = c(s_0) + \sqrt{R^2 - s^2}, \quad 0 \leq s \leq s_0.$$

For $s < s_0$, monotonicity of $c(s)$ gives

$$y_{\text{arc}}(s) > c(s) + \sqrt{R^2 - s^2} = f(s).$$

Thus the circular portion lies above the entrance and exit boundaries, inside the field. Before entry, the incoming line lies below the boundary because

$$y_{\text{line}}(s) = (a - s) \tan \theta < (a - s) \frac{s}{\sqrt{R^2 - s^2}} = f(s)$$

for $s > s_0$ in the boundary's domain. Reflection establishes the same result after exit. The boundary therefore produces the required focusing.

The solution was checked against the official solution.