

IPhO 1974

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Problem 1: Inelastic collision of two hydrogen atoms (Bohr model, Doppler shift)

A hydrogen atom in the ground state, moving with velocity v , collides with another hydrogen atom in the ground state at rest. Using the Bohr model find the smallest velocity v_0 of the atom below which the collision must be elastic.

At velocity v_0 the collision may be inelastic and the colliding atoms may emit electromagnetic radiation. Estimate the difference of frequencies of the radiation emitted in the direction of the initial velocity of the hydrogen atom and in the opposite direction as a fraction (expressed in percents) of their arithmetic mean value.

Data:

$$E_i = \frac{me^4}{2\hbar^2} = 13.6 \text{ eV} = 2.18 \cdot 10^{-18} \text{ J}; \quad (\text{ionization energy of hydrogen atom})$$

$$m_H = 1.67 \cdot 10^{-27} \text{ kg}; \quad (\text{mass of hydrogen atom})$$

(m - mass of electron; e - electric charge of electron; $\hbar$ - Planck constant; numerical values of these quantities are not necessary.)

Official English text of the problem.

Answers

The original has two unnumbered questions and gives no point values.

- **First question — threshold speed:**

$$v_0 = \sqrt{\frac{3E_i}{m_H}} \simeq 6.26 \times 10^4 \text{ m s}^{-1}.$$

- **Second question — frequency difference:** With ν_+ denoting forward radiation and ν_- backward radiation,

$$\frac{\nu_+ - \nu_-}{(\nu_+ + \nu_-)/2} \simeq \frac{v_0}{c} = 2.09 \times 10^{-4} = 0.0209\%.$$

The forward radiation has the higher frequency.

Solution

First question: threshold for an inelastic collision

In the Bohr model, the hydrogen energy levels are

$$E_n = -\frac{E_i}{n^2}.$$

The smallest possible atomic excitation takes one atom from $n = 1$ to $n = 2$, requiring

$$\Delta E = E_2 - E_1 = E_i \left(1 - \frac{1}{4}\right) = \frac{3}{4}E_i = 10.2 \text{ eV}.$$

Both energy and momentum must be conserved. In particular, the entire initial kinetic energy cannot become excitation energy, because the atoms must retain the initial total momentum.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be their velocities after the collision. Neglecting the tiny change in atomic mass upon excitation, momentum conservation gives

$$m_H \mathbf{v} = m_H \mathbf{v}_1 + m_H \mathbf{v}_2.$$

Their final translational kinetic energy can therefore be written as

$$K_f = \frac{1}{2}m_H(v_1^2 + v_2^2) = \frac{1}{4}m_H v^2 + \frac{1}{4}m_H |\mathbf{v}_1 - \mathbf{v}_2|^2.$$

Its minimum is

$$K_{f,\min} = \frac{1}{4}m_H v^2,$$

attained when both atoms move together with velocity $\mathbf{v}/2$.

Thus the maximum energy available for excitation is

$$K_i - K_{f,\min} = \frac{1}{2}m_H v^2 - \frac{1}{4}m_H v^2 = \frac{1}{4}m_H v^2.$$

The excitation threshold consequently satisfies

$$\frac{1}{4}m_H v_0^2 = \Delta E = \frac{3}{4}E_i,$$

so

$$v_0 = \sqrt{\frac{3E_i}{m_H}}.$$

Using $E_i = 2.18 \times 10^{-18} \text{ J}$, consistent with the given 13.6 eV,

$$v_0 = \sqrt{\frac{3(2.18 \times 10^{-18})}{1.67 \times 10^{-27}}} \simeq \boxed{6.26 \times 10^4 \text{ m s}^{-1}}.$$

Below this speed, no atomic excitation is energetically possible, so the collision is elastic within the intended Bohr-level model.

Second question: Doppler frequency difference

At threshold, the atoms have no relative kinetic energy after excitation. The excited atom therefore moves in the initial direction with speed

$$u = \frac{v_0}{2} \simeq 3.13 \times 10^4 \text{ m s}^{-1}.$$

When it returns to the ground state, its rest-frame emission frequency is

$$\nu_* = \frac{\Delta E}{h}, \quad h = 2\pi\hbar,$$

neglecting photon recoil.

Since $u/c \simeq 1.04 \times 10^{-4} \ll 1$, the first-order Doppler formulas suffice:

$$\nu_+ \simeq \nu_* \left(1 + \frac{u}{c}\right), \quad \nu_- \simeq \nu_* \left(1 - \frac{u}{c}\right).$$

Hence

$$\nu_+ - \nu_- \simeq 2\nu_* \frac{u}{c}, \quad \frac{\nu_+ + \nu_-}{2} \simeq \nu_*.$$

Dividing eliminates the rest-frame frequency:

$$\frac{\nu_+ - \nu_-}{(\nu_+ + \nu_-)/2} \simeq \frac{2u}{c} = \frac{v_0}{c}.$$

With $c \simeq 3.00 \times 10^8 \text{ m s}^{-1}$, the requested percentage is

$$\frac{6.26 \times 10^4}{3.00 \times 10^8} \times 100\% \simeq \boxed{0.0209\%}.$$

The solution was checked against the official solution.

Problem 2: Plate with a refractive index varying across its width

Consider a parallel, transparent plate of thickness d – Fig. 1. Its refraction index varies as

$$n = \frac{n_0}{1 - \frac{x}{R}}.$$

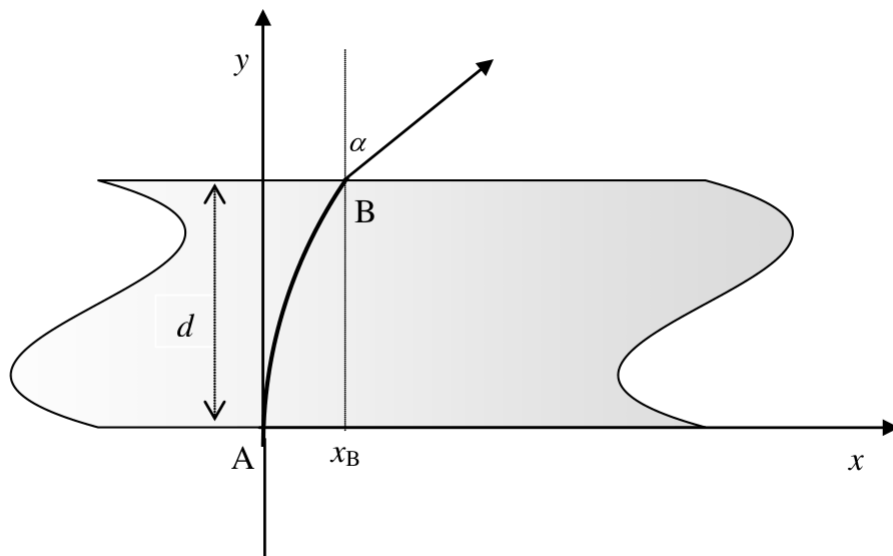


Fig. 1

A light beam enters from the air perpendicularly to the plate at the point A ($x_A = 0$) and emerges from it at the point B at an angle α .

1. Find the refraction index n_B at the point B.
2. Find x_B (i.e. value of x at the point B)
3. Find the thickness d of the plate.

Data:

$$n_0 = 1.2; \quad R = 13 \text{ cm}; \quad \alpha = 30^\circ.$$

Official English text of the problem.

Answers

$$1. n_B = \sqrt{n_0^2 + \sin^2 \alpha} = \boxed{1.3} \text{ (dimensionless).}$$

$$2. x_B = R \left(1 - \frac{n_0}{\sqrt{n_0^2 + \sin^2 \alpha}} \right) = \boxed{1.0 \text{ cm}}.$$

$$3. d = \frac{R \sin \alpha}{\sqrt{n_0^2 + \sin^2 \alpha}} = \boxed{5.0 \text{ cm}}.$$

Solution

1. Refraction index at B

We use geometrical optics, treating the refractive index as varying smoothly on the scale of the wavelength, and take the refractive index of air to be 1.

Let β denote the angle the ray inside the plate makes with the positive y -axis, which is normal to the plate surfaces. Normal incidence at A gives $\beta_A = 0$.

To determine how the ray bends, imagine dividing the plate into infinitesimally thin vertical layers of constant refractive index. The normal to each layer interface points along x , so the ray's angle to that normal is $90^\circ - \beta$. Snell's law across successive layers therefore gives

$$n \sin(90^\circ - \beta) = n \cos \beta = \text{constant}.$$

Evaluating this constant at A, where $n = n_0$, yields

$$\boxed{n(x) \cos \beta = n_0}.$$

At B, let the internal incidence angle be β_B . Thus

$$n_B \cos \beta_B = n_0.$$

Snell's law at the upper surface, where the ray emerges into air, gives

$$n_B \sin \beta_B = \sin \alpha.$$

Squaring and adding these equations eliminates β_B :

$$n_B^2 (\cos^2 \beta_B + \sin^2 \beta_B) = n_0^2 + \sin^2 \alpha.$$

Hence

$$n_B = \sqrt{n_0^2 + \sin^2 \alpha}.$$

Numerically,

$$n_B = \sqrt{1.2^2 + \sin^2 30^\circ} = \sqrt{1.44 + 0.25} = \boxed{1.3}.$$

2. Position x_B

Using the prescribed index profile at B,

$$n_B = \frac{n_0}{1 - x_B/R},$$

we obtain

$$x_B = R \left(1 - \frac{n_0}{n_B} \right).$$

Substitution gives

$$x_B = 13 \text{ cm} \left(1 - \frac{1.2}{1.3} \right) = 13 \text{ cm} \left(\frac{1}{13} \right) = \boxed{1.0 \text{ cm}}.$$

3. Thickness d

Combining the conserved quantity with the index profile gives

$$\cos \beta = \frac{n_0}{n(x)} = 1 - \frac{x}{R}.$$

The refractive index increases with x , so the ray bends towards positive x . On this branch, its tangent satisfies

$$\frac{dy}{dx} = \cot \beta = \frac{\cos \beta}{\sqrt{1 - \cos^2 \beta}} = \frac{R - x}{\sqrt{2Rx - x^2}}.$$

Integrating from A, where $x = y = 0$, to an arbitrary point on the ray,

$$y(x) = \int_0^x \frac{R - u}{\sqrt{2Ru - u^2}} du = \sqrt{2Rx - x^2}.$$

Consequently, the ray follows the circular arc

$$(x - R)^2 + y^2 = R^2,$$

with centre $(R, 0)$ and radius R . The relevant arc starts at A with a vertical tangent and bends to the right.

At B, $y = d$, so

$$d = \sqrt{2Rx_B - x_B^2}.$$

Equivalently, substituting the expression for x_B ,

$$d = R \sqrt{1 - \left(\frac{n_0}{n_B} \right)^2} = \boxed{\frac{R \sin \alpha}{\sqrt{n_0^2 + \sin^2 \alpha}}}.$$

Numerically,

$$d = \sqrt{2(13 \text{ cm})(1.0 \text{ cm}) - (1.0 \text{ cm})^2} = \sqrt{25 \text{ cm}^2} = \boxed{5.0 \text{ cm}}.$$

The solution was checked against the official solution.

Problem 3: Engine using a gas source and semipermeable membranes

A scientific expedition stayed on an uninhabited island. The members of the expedition had had some sources of energy, but after some time these sources exhausted. Then they decided to construct an alternative energy source. Unfortunately, the island was very quiet: there were no winds, clouds uniformly covered the sky, the air pressure was constant and the temperatures of air and water in the sea were the same during day and night. Fortunately, they found a source of chemically neutral gas outgoing very slowly from a cavity. The pressure and temperature of the gas are exactly the same as the pressure and temperature of the atmosphere.

The expedition had, however, certain membranes in its equipment. One of them was ideally transparent for gas and ideally non-transparent for air. Another one had an opposite property: it was ideally transparent for air and ideally non-transparent for gas. The members of the expedition had materials and tools that allowed them to make different mechanical devices such as cylinders with pistons, valves etc. They decided to construct an engine by using the gas from the cavity.

Show that there is no theoretical limit on the power of an ideal engine that uses the gas and the membranes considered above.

Official English text of the problem.

Answers

- **Problem 3:** Let $\dot{n}_g > 0$ be the molar flow of source gas, T_0 the ambient temperature, and r the number of moles of air used per mole of gas. An ideal membrane engine can deliver

$$\mathcal{P}(r) = \dot{n}_g R T_0 \left[\ln(1+r) + r \ln\left(1 + \frac{1}{r}\right) \right].$$

Since $\mathcal{P}(r) \rightarrow \infty$ as $r \rightarrow \infty$, there is **no finite theoretical upper bound on its power**, even at a fixed, small gas flow.

Solution

Denote the common atmospheric and source pressure by p_0 , and their temperature by T_0 . Use the ideal-gas approximation, neglect friction, and allow isothermal operation through heat exchange with the surroundings. The available resource is the initially separate state of the source gas and atmospheric air: the membranes allow their mixing to produce mechanical work.

Consider a charge consisting of n moles of source gas and rn moles of air, initially at p_0, T_0 . Their volumes are

$$V_g = \frac{nRT_0}{p_0}, \quad V_a = \frac{rnRT_0}{p_0} = rV_g,$$

where R is the universal gas constant.

Place these gases in a cylinder of fixed total volume

$$V = V_g + V_a,$$

with source gas on the left and air on the right. Initially, two adjacent membrane pistons separate them:

- The left membrane passes source gas but blocks air.
- The right membrane passes air but blocks source gas.

Allow the left membrane to move toward the left end and the right membrane toward the right end, each driving a mechanical load. Between the membranes there is a mixture; outside them there is pure source gas on the left and pure air on the right. Ultimately, both components occupy the entire cylinder.

The essential property is that **a permeating component has equal partial pressures on the two sides of its membrane** in reversible operation. Thus, across the air-permeable piston, the forces due to air cancel. Only the source-gas partial pressure drives that piston. If the volume accessible to the source gas is v_g , its work is

$$dW_g = p_g dv_g = \frac{nRT_0}{v_g} dv_g.$$

Similarly, the gas-permeable piston extracts work from the expanding air:

$$dW_a = p_a dv_a = \frac{rnRT_0}{v_a} dv_a.$$

Consequently, the total work obtained in reversible isothermal mixing is

$$\begin{aligned} W &= \int_{V_g}^V \frac{nRT_0}{v_g} dv_g + \int_{V_a}^V \frac{rnRT_0}{v_a} dv_a \\ &= nRT_0 \ln \frac{V}{V_g} + rnRT_0 \ln \frac{V}{V_a} \\ &= \boxed{nRT_0 \left[\ln(1+r) + r \ln \left(1 + \frac{1}{r} \right) \right]}. \end{aligned}$$

At the end, the mixture has total pressure

$$p_f = \frac{(1+r)nRT_0}{V} = p_0.$$

It can therefore be exhausted and replaced with fresh gas and air at the common pressure p_0 . Ordinary pistons and bypass valves can perform this exchange and reset the membranes. Since the incoming and outgoing total volumes are equal, their pressure-volume work cancels:

$$W_{\text{intake/exhaust}} = p_0(V_g + V_a) - p_0V = 0.$$

The spent mixture is discharged, so the engine does not need to separate it again. Thus W is useful work per charge of n moles of source gas.

For a steady gas supply $\dot{n}_g$, operating successive or parallel cylinders gives the ideal mean power

$$\mathcal{P}(r) = \frac{\dot{n}_g}{n} W = \dot{n}_g R T_0 \left[\ln(1+r) + r \ln\left(1 + \frac{1}{r}\right) \right].$$

Equivalently, if q_g is the source's volume flow measured at p_0, T_0 , then

$$\mathcal{P}(r) = p_0 q_g \left[\ln(1+r) + r \ln\left(1 + \frac{1}{r}\right) \right],$$

with $p_0 q_g$ measured in watts.

Now increase the amount of air used per mole of source gas. Since

$$\ln(1+r) \longrightarrow \infty, \quad r \ln\left(1 + \frac{1}{r}\right) \longrightarrow 1,$$

we obtain

$$\mathcal{P}(r) = \dot{n}_g R T_0 [\ln r + 1 + o(1)] \longrightarrow \infty.$$

Hence even an arbitrarily small, fixed, positive gas flow imposes no finite theoretical power ceiling.

This conclusion allows unrestricted apparatus size, air throughput, membrane area and heat exchange. Slow, nearly reversible strokes can be operated in sufficiently many parallel cylinders to process the prescribed gas flow. Increasing the power requires an increasingly large amount of fresh air per mole of gas.

Finally, this respects the second law. For ideal gases at unchanged temperature, the mixing process leaves internal energy unchanged, so the absorbed heat equals the work:

$$Q = W.$$

The entropy increase of the gases is

$$\Delta S_{\text{mix}} = nR \ln \frac{V}{V_g} + rnR \ln \frac{V}{V_a} = \frac{W}{T_0},$$

which balances the heat reservoir's entropy decrease in reversible operation. The engine consumes the initial separation of gas and air. The divergence assumes access to arbitrarily much fresh air; finite reservoirs and finite equipment impose practical limits.

The solution was checked against the official solution.