

IPhO 1972

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Problem 1: Three cylinders on an inclined plane

Three cylinders with the same mass, the same length and the same external radius are initially resting on an inclined plane. The coefficient of sliding friction on the inclined plane, μ , is known and has the same value for all the cylinders. The first cylinder is empty (tube), the second is homogeneous filled, and the third has a cavity exactly like the first, but closed with two negligible mass lids and filled with a liquid with the same density like the cylinder's walls. The friction between the liquid and the cylinder wall is considered negligible. The density of the material of the first cylinder is n times greater than that of the second or of the third cylinder.

Determine:

- The linear acceleration of the cylinders in the non-sliding case. Compare all the accelerations.
- Condition for angle α of the inclined plane so that no cylinders is sliding.
- The reciprocal ratios of the angular accelerations in the case of roll over with sliding of all the three cylinders. Make a comparison between these accelerations.
- The interaction force between the liquid and the walls of the cylinder in the case of sliding of this cylinder, knowing that the liquid mass is m_l .

Official English text of the problem.

Answers

Let R be the common external radius and β_i the angular acceleration of the solid part of cylinder i . Let $\hat{e}_{\parallel}$ point downhill and $\hat{e}_{\perp}$ point normally away from the plane. The comparisons below assume genuine cavities, $n > 1$, and $0 < \alpha < \pi/2$.

- a)** The accelerations downhill are

$$a_1 = \frac{2n}{4n-1}g \sin \alpha, \quad a_2 = \frac{2}{3}g \sin \alpha, \quad a_3 = \frac{2n^2}{2n^2+2n-1}g \sin \alpha$$

and

$$a_1 < a_2 < a_3 < g \sin \alpha.$$

- **b)** If the given μ also represents the limiting static-friction coefficient,

$$\tan \alpha \leq \mu \frac{4n-1}{2n-1}.$$

Cylinder 1 sets the limiting condition. If μ denotes strictly kinetic friction, the static-friction coefficients are additionally needed for an exact no-sliding condition.

- **c)** For nonzero sliding friction,

$$\beta_1 : \beta_2 : \beta_3 = n : (2n-1) : n^2, \quad \beta_1 < \beta_2 < \beta_3.$$

Equivalently, the inverse accelerations satisfy

$$\frac{1}{\beta_1} : \frac{1}{\beta_2} : \frac{1}{\beta_3} = \frac{1}{n} : \frac{1}{2n-1} : \frac{1}{n^2}.$$

- **d)** The resultant force exerted by the walls on the liquid is

$$\mathbf{F}_{\text{walls} \rightarrow l} = m_l g \cos \alpha (\hat{\mathbf{e}}_{\perp} - \mu \hat{\mathbf{e}}_{\parallel}).$$

Hence the interaction-force magnitude is

$$F = m_l g \cos \alpha \sqrt{1 + \mu^2}.$$

The force on the liquid makes an angle $\arctan \mu$ with the outward normal, towards uphill. The force on the walls is equal and opposite.

Solution

Let the common total mass be M , the common length be L , and the cavity radius of cylinders 1 and 3 be r . Denote the density of cylinder 2, the walls of cylinder 3, and its liquid by ρ . Cylinder 1 has density $n\rho$.

Equality of the masses gives

$$M = \rho \pi R^2 L = n \rho \pi (R^2 - r^2) L.$$

Therefore,

$$\frac{r^2}{R^2} = \frac{n-1}{n}.$$

The mass of the third cylinder's shell and its liquid mass are consequently

$$M_{\text{shell},3} = \frac{M}{n}, \quad m_l = M \frac{n-1}{n}.$$

We treat the cylinders as rigid, neglect rolling resistance, and model the completely enclosed liquid as incompressible and inviscid. Rotation of the circular wall does not impose tangential motion on this liquid. Pressure forces exert no torque about the cylinder axis, so the initially nonrotating liquid translates with the cylinder without spinning.

Thus the entire mass M contributes to translation, but only the shell contributes to cylinder 3's rotational inertia. Using the axial moment of inertia $\frac{1}{2}m(R^2 + r^2)$ of a uniform annular cylinder,

$$I_1 = \frac{1}{2}M(R^2 + r^2) = MR^2 \frac{2n-1}{2n},$$

$$I_2 = \frac{1}{2}MR^2,$$

$$I_3 = \frac{1}{2} \frac{M}{n}(R^2 + r^2) = MR^2 \frac{2n-1}{2n^2}.$$

Here I_3 denotes the inertia of the rotating shell.

Define the dimensionless inertias

$$k_i = \frac{I_i}{MR^2}, \quad k_1 = \frac{2n-1}{2n}, \quad k_2 = \frac{1}{2}, \quad k_3 = \frac{2n-1}{2n^2}.$$

a) Linear accelerations without sliding

Let f_i be the static-friction force, directed uphill. Newton's second law along the plane and the rotational equation about the cylinder axis give

$$Ma_i = Mg \sin \alpha - f_i, \quad I_i \beta_i = f_i R.$$

For rolling without slipping,

$$a_i = R\beta_i.$$

Therefore,

$$f_i = \frac{I_i}{R^2} a_i = Mk_i a_i,$$

and hence

$$a_i = \frac{g \sin \alpha}{1 + k_i}.$$

Substituting the three inertias,

$$a_1 = \frac{2n}{4n-1} g \sin \alpha, \quad a_2 = \frac{2}{3} g \sin \alpha, \quad a_3 = \frac{2n^2}{2n^2 + 2n - 1} g \sin \alpha.$$

For $n > 1$,

$$k_1 - k_2 = \frac{n-1}{2n} > 0, \quad k_2 - k_3 = \frac{(n-1)^2}{2n^2} > 0.$$

Since a_i decreases as k_i increases,

$$a_1 < a_2 < a_3 < g \sin \alpha.$$

The liquid-filled cylinder accelerates fastest because its liquid adds translating mass without adding rotating inertia.

b) Condition for all cylinders to roll without sliding

There is no acceleration normal to the plane, so each cylinder has normal reaction

$$N = Mg \cos \alpha.$$

The static friction required for rolling is

$$f_i = Mg \sin \alpha \frac{k_i}{1 + k_i},$$

giving

$$\frac{f_i}{N} = \frac{k_i}{1 + k_i} \tan \alpha.$$

To distinguish static and sliding friction explicitly, let $\mu_{s,i}$ denote the limiting static-friction coefficient for cylinder i . All three can roll without slipping precisely when

$$\tan \alpha \leq \min \left\{ \mu_{s,1} \frac{4n-1}{2n-1}, 3\mu_{s,2}, \mu_{s,3} \frac{2n^2+2n-1}{2n-1} \right\}.$$

Under the usual single-coefficient idealization, $\mu_{s,1} = \mu_{s,2} = \mu_{s,3} = \mu$. Since $k_1 > k_2 > k_3$, cylinder 1 requires the largest friction relative to its normal reaction and therefore sets the bound:

$$\tan \alpha \leq \mu \frac{4n-1}{2n-1}, \quad \text{or} \quad \alpha \leq \arctan \left(\mu \frac{4n-1}{2n-1} \right).$$

Equality corresponds to limiting static friction for cylinder 1.

c) Angular-acceleration ratios when all three slide

In the stipulated regime, the cylinders slip downhill relative to the plane. Kinetic friction acts uphill and has the same magnitude for each:

$$f = \mu Mg \cos \alpha.$$

Thus their translational accelerations are equal:

$$a_{sl} = g(\sin \alpha - \mu \cos \alpha).$$

Their angular accelerations follow from

$$I_i \beta_i = fR,$$

without imposing the no-slip relation $a = R\beta$. Consequently,

$$\beta_1 = \frac{2n}{2n-1} \frac{\mu g \cos \alpha}{R}, \quad \beta_2 = 2 \frac{\mu g \cos \alpha}{R}, \quad \beta_3 = \frac{2n^2}{2n-1} \frac{\mu g \cos \alpha}{R}.$$

Removing the common factor gives

$$\beta_1 : \beta_2 : \beta_3 = n : (2n-1) : n^2.$$

The mutual ratios are

$$\frac{\beta_2}{\beta_1} = \frac{2n-1}{n}, \quad \frac{\beta_3}{\beta_2} = \frac{n^2}{2n-1}, \quad \frac{\beta_3}{\beta_1} = n.$$

For $n > 1$,

$$2n-1 > n, \quad n^2 - (2n-1) = (n-1)^2 > 0,$$

so

$$\beta_1 < \beta_2 < \beta_3.$$

As a consistency check, downhill slipping from rest requires $a_{sl} - R\beta_i > 0$. In the single-coefficient model, all three satisfy this when

$$\tan \alpha > \mu \frac{2n^2+2n-1}{2n-1}.$$

Cylinder 3 is the last to begin slipping as the inclination increases.

d) Interaction force between the liquid and the walls

The liquid's centre of mass accelerates downhill with the sliding cylinder:

$$a_{sl} = g(\sin \alpha - \mu \cos \alpha).$$

Let $F_{\parallel}$ and $F_{\perp}$ be the components of the resultant force exerted by the walls on the liquid, positive downhill and normally away from the plane. Applying Newton's second law to the liquid alone,

$$F_{\parallel} + m_l g \sin \alpha = m_l a_{sl},$$

$$F_{\perp} - m_l g \cos \alpha = 0.$$

Hence

$$F_{\parallel} = -\mu m_l g \cos \alpha, \quad F_{\perp} = m_l g \cos \alpha.$$

These components arise from the resultant pressure force; an uphill component does not require viscous friction between the liquid and the wall. Therefore,

$$\mathbf{F}_{\text{walls} \rightarrow l} = m_l g \cos \alpha (\hat{\mathbf{e}}_{\perp} - \mu \hat{\mathbf{e}}_{\parallel})$$

and

$$F = \sqrt{F_{\parallel}^2 + F_{\perp}^2} = m_l g \cos \alpha \sqrt{1 + \mu^2}.$$

Its inclination from the outward normal towards uphill is

$$\theta = \arctan \mu.$$

By Newton's third law, the liquid exerts the opposite force on the walls.

The solution was checked against the official solution.

Problem 2: Argon and oxygen in two cylinders with connected pistons

Two cylinders A and B, with equal diameters have inside two pistons with negligible mass connected by a rigid rod. The pistons can move freely. The rod is a short tube with a valve. The valve is initially closed (fig. 2.1).

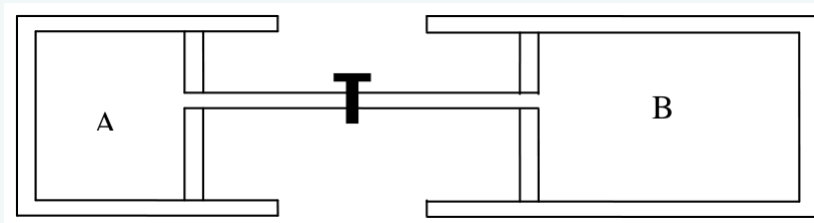


Fig. 2.1

The cylinder A and his piston is adiabatically insulated and the cylinder B is in thermal contact with a thermostat which has the temperature $\theta = 27^\circ\text{C}$.

Initially the piston of the cylinder A is fixed and inside there is a mass $m = 32 \text{ kg}$ of argon at a pressure higher than the atmospheric pressure. Inside the cylinder B there is a mass of oxygen at the normal atmospheric pressure.

Liberating the piston of the cylinder A, it moves slowly enough (quasi-static) and at equilibrium the volume of the gas is eight times higher, and in the cylinder B the oxygen's density increased two times. Knowing that the thermostat received the heat $Q' = 747,9 \cdot 10^4 \text{ J}$, determine:

- Establish on the base of the kinetic theory of the gases, studying the elastic collisions of the molecules with the piston, that the thermal equation of the process taking place in the cylinder A is $TV^{2/3} = \text{constant}$.
- Calculate the parameters p , V , and T of argon in the initial and final states.
- Opening the valve which separates the two cylinders, calculate the final pressure of the mixture of the gases.

The kilo-molar mass of argon is $\mu = 40 \text{ kg/kmol}$.

Official English text of the problem.

Answers

The following are the formal results from the requested adiabatic relation, final mechanical equilibrium, and overall energy conservation. The quasistatic wording has a physical consistency issue: on release the argon is at $64p_0$ but the oxygen only at p_0 , so the massless piston assembly cannot move quasistatically.

- a)** $TV^{2/3} = \text{constant}$, equivalently $pV^{5/3} = \text{constant}$.
- b)** The argon states, before opening the valve, are:

State	Pressure p	Volume V	Temperature T
Initial	$64p_0 \simeq 6.48 \times 10^6 \text{ Pa}$	1.03 m^3	$1.00 \times 10^3 \text{ K}$
Final	$2p_0 \simeq 2.03 \times 10^5 \text{ Pa}$	8.20 m^3	250 K

- c)** After opening the valve and reaching thermal equilibrium,

$$p_{\text{mix}} = \frac{166}{75} p_0 \simeq 2.24 \times 10^5 \text{ Pa}.$$

Here p_0 is normal atmospheric pressure.

Solution

Treat both gases as ideal and neglect the volume of the short connecting tube. Use

$$p_0 = 1.01325 \times 10^5 \text{ Pa}, \quad T_0 \simeq 300 \text{ K}, \quad R = 8.31 \times 10^3 \text{ J kmol}^{-1} \text{ K}^{-1}.$$

The amount of argon is

$$n_A = \frac{m}{\mu} = \frac{32}{40} = 0.800 \text{ kmol}.$$

Subscripts 1 and 2 denote the initial state and the equilibrium state reached with the valve still closed.

a) Derivation from molecular collisions

Let the argon piston have area S and move outward with speed u , small compared with molecular thermal speeds. Consider an argon atom of mass m_0 approaching it with velocity component v_x normal to the piston.

In the instantaneous rest frame of the piston, an elastic collision reverses the normal velocity. Thus, in the laboratory frame,

$$v'_x = 2u - v_x.$$

The other velocity components remain unchanged. The atom's energy change is therefore

$$\Delta\varepsilon = \frac{m_0}{2} [(2u - v_x)^2 - v_x^2] = -2m_0u(v_x - u).$$

The impulse delivered to the piston is

$$I = 2m_0(v_x - u),$$

so

$$\Delta\varepsilon = -uI.$$

Summing over molecular collisions, the impulse delivered per unit time is the force $p_A S$. Consequently,

$$\frac{dU_A}{dt} = -p_A S u = -p_A \frac{dV_A}{dt},$$

or

$$dU_A = -p_A dV_A.$$

For an equilibrium monatomic ideal gas, kinetic theory gives

$$p_A = \frac{Nm_0}{V_A} \langle v_x^2 \rangle = \frac{Nm_0}{3V_A} \langle v^2 \rangle, \quad U_A = \frac{1}{2} Nm_0 \langle v^2 \rangle.$$

Hence

$$p_A V_A = \frac{2}{3} U_A, \quad U_A = \frac{3}{2} N k_B T_A.$$

The slow-motion assumption allows intermolecular collisions to maintain an isotropic equilibrium velocity distribution.

Substitution into the energy equation yields

$$\frac{3}{2} N k_B dT_A = -\frac{N k_B T_A}{V_A} dV_A.$$

Therefore

$$\frac{dT_A}{T_A} = -\frac{2}{3} \frac{dV_A}{V_A}.$$

Integrating,

$$T_A V_A^{2/3} = \text{constant}.$$

Combining this with $p_A V_A = n_A R T_A$ also gives

$$p_A V_A^{5/3} = \text{constant}.$$

b) Initial and final argon states

The specified argon expansion gives

$$V_{A2} = 8V_{A1}.$$

Using part a),

$$\frac{T_{A2}}{T_{A1}} = \left(\frac{V_{A1}}{V_{A2}} \right)^{2/3} = 8^{-2/3} = \frac{1}{4}.$$

Thus

$$T_{A2} = \frac{1}{4}T_{A1}.$$

The oxygen mass is unchanged before the valve opens. Its doubled density therefore means

$$V_{B2} = \frac{1}{2}V_{B1}.$$

At both equilibrium endpoints, oxygen is at the thermostat temperature T_0 . Its ideal-gas equation then gives

$$p_{B2} = p_{B1} \frac{V_{B1}}{V_{B2}} = 2p_0.$$

At final mechanical equilibrium, the forces on the connected piston assembly balance. The atmospheric forces cancel because the piston areas are equal, leaving

$$p_{A2} = p_{B2} = 2p_0.$$

The argon adiabatic relation now gives

$$p_{A1} = p_{A2} \left(\frac{V_{A2}}{V_{A1}} \right)^{5/3} = 2p_0 8^{5/3} = \boxed{64p_0}.$$

To determine the temperatures, consider both gases together. Equal piston displacements and equal cross-sectional areas imply

$$dV_A = -dV_B.$$

Thus the combined gas volume is constant, and the net work against the atmosphere is zero. Work transmitted through the rod is internal to the combined system. Since the pistons are at rest at both endpoints, overall energy conservation gives

$$\Delta U_A + \Delta U_B = -Q'.$$

Oxygen has the same initial and final temperature, so, for an ideal gas,

$$\Delta U_B = 0.$$

Therefore

$$Q' = \frac{3}{2}n_A R(T_{A1} - T_{A2}) = \frac{9}{8}n_A R T_{A1}.$$

With $Q' = 7.479 \times 10^6$ J,

$$T_{A1} = \frac{8Q'}{9n_A R} = \frac{8(7.479 \times 10^6)}{9(0.800)(8.31 \times 10^3)} = \boxed{1000 \text{ K}},$$

and

$$T_{A2} = 250 \text{ K}.$$

Finally,

$$V_{A1} = \frac{n_A R T_{A1}}{p_{A1}} = \frac{(0.800)(8.31 \times 10^3)(1000)}{64(1.01325 \times 10^5)} = 1.025 \text{ m}^3,$$

and

$$V_{A2} = 8V_{A1} = 8.201 \text{ m}^3.$$

To three significant figures,

$$(p_{A1}, V_{A1}, T_{A1}) = (6.48 \times 10^6 \text{ Pa}, 1.03 \text{ m}^3, 1.00 \times 10^3 \text{ K})$$

and

$$(p_{A2}, V_{A2}, T_{A2}) = (2.03 \times 10^5 \text{ Pa}, 8.20 \text{ m}^3, 250 \text{ K}).$$

c) Final pressure after opening the valve

The piston geometry gives

$$V_{A2} - V_{A1} = V_{B1} - V_{B2}.$$

Using $V_{A2} = 8V_{A1}$ and $V_{B2} = V_{B1}/2$,

$$7V_{A1} = \frac{1}{2}V_{B1}.$$

Hence

$$V_{B1} = 14V_{A1}, \quad V_{B2} = 7V_{A1},$$

and the constant total volume is

$$V_{\text{tot}} = 15V_{A1}.$$

After the valve opens, the gases mix and eventually reach the thermostat temperature T_0 . Although cylinder A is insulated, its gas can exchange energy through the open connection with cylinder B.

The conserved amount of oxygen satisfies

$$n_B R T_0 = p_0 V_{B1} = 14p_0 V_{A1}.$$

For argon,

$$n_A R T_0 = p_{A1} V_{A1} \frac{T_0}{T_{A1}} = 64p_0 V_{A1} \frac{300}{1000} = 19.2p_0 V_{A1}.$$

Applying the ideal-gas equation to the final mixture,

$$p_{\text{mix}} = \frac{(n_A + n_B) R T_0}{V_{\text{tot}}} = \frac{(19.2 + 14)p_0 V_{A1}}{15V_{A1}}.$$

Therefore

$$p_{\text{mix}} = \frac{166}{75} p_0 = 2.2133 p_0 \simeq 2.24 \times 10^5 \text{ Pa}.$$

The solution was checked against the official solution.

Problem 3: Dielectric liquid rising between the plates of a charged capacitor

A plane capacitor with rectangular plates is fixed in a vertical position having the lower part in contact with a dielectric liquid (fig. 3.1)

Determine the height, h , of the liquid between the plates and explain the phenomenon.

The capillarity effects are neglected.

It is supposed that the distance between the plates is much smaller than the linear dimensions of the plates.

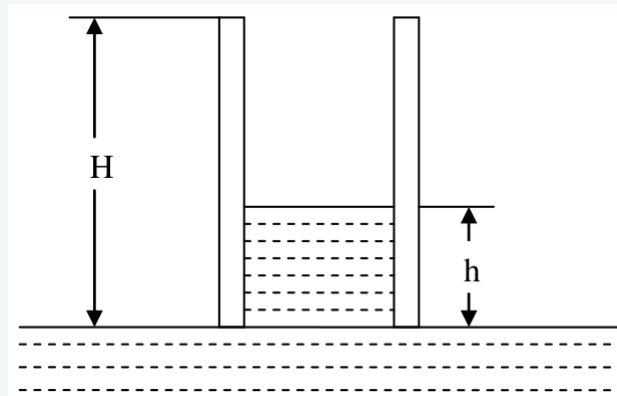


Fig. 3.1

It is known: the initial intensity of the electric field of the charged capacitor, E , the density ρ , the relative electric permittivity ε_r of the liquid, and the height H of the plates of the capacitor.

Discussion.

Official English text of the problem.

Answers

- **Problem 3 — Height:** Assuming that the charged capacitor is isolated, put $a = \varepsilon_r - 1 > 0$. Let h_* be the unique nonnegative solution of

$$h_* \left(1 + a \frac{h_*}{H} \right)^2 = \frac{\varepsilon_0 a E^2}{2\rho g}.$$

The equilibrium height is $h = h_*$ when $h_* < H$. The liquid fills the gap to the top, $h = H$ in the ideal approximation, when

$$E \geq E_c = \varepsilon_r \sqrt{\frac{2\rho g H}{\varepsilon_0(\varepsilon_r - 1)}}.$$

- **Problem 3 — Explanation and discussion:** Polarization produces an upward electric force. At a partial-rise equilibrium, its pressure balances the hydrostatic pressure:

$$\rho g h = \frac{1}{2} \varepsilon_0 (\varepsilon_r - 1) \left[\frac{E}{1 + (\varepsilon_r - 1)h/H} \right]^2.$$

This equilibrium is unique and stable. There is no rise if $E = 0$ or $\varepsilon_r = 1$.

- **Problem 3 — Fixed-voltage alternative:** If a source maintains the initial voltage instead, the result is

$$h = \min \left\{ H, \frac{\varepsilon_0 (\varepsilon_r - 1) E^2}{2 \rho g} \right\}.$$

Solution

Problem 3. The reference to the *initial* field, together with the absence of a specified voltage source, suggests an isolated capacitor, whose free charge remains constant. We first solve this case.

Let the plate width be b , their separation be d , and the external liquid surface be at $z = 0$. We treat the reservoir as sufficiently large that this level remains fixed. The liquid is a linear, homogeneous, incompressible dielectric; air has permittivity ε_0 . Fringing corrections to capacitance are neglected because d is much smaller than the plate dimensions.

Capacitance and field during the rise. When the liquid has risen to height $h < H$, the liquid-filled and air-filled portions form two capacitors in parallel. Thus

$$C(h) = \frac{\varepsilon_0 b (H - h)}{d} + \frac{\varepsilon_0 \varepsilon_r b h}{d} = C_0 \left(1 + a \frac{h}{H} \right),$$

where

$$C_0 = \frac{\varepsilon_0 b H}{d}, \quad a = \varepsilon_r - 1.$$

Initially $V_0 = Ed$, so the conserved charge is

$$Q = C_0 Ed.$$

Consequently, the field after the rise is

$$\mathcal{E}(h) = \frac{Q}{C(h)d} = \frac{E}{1 + ah/H}.$$

The field therefore decreases as liquid enters the isolated capacitor.

Equilibrium height. At fixed charge, the electric energy is

$$U_e(h) = \frac{Q^2}{2C(h)} = \frac{\varepsilon_0 b d H E^2}{2(1 + ah/H)}.$$

The gravitational energy required to raise liquid from the reservoir into the gap is

$$U_g(h) = \int_0^h \rho b d g z dz = \frac{1}{2} \rho b d g h^2.$$

A static equilibrium minimizes the total energy. Hence

$$0 = \frac{d}{dh}(U_e + U_g) = bd \left[\rho gh - \frac{\varepsilon_0 a E^2}{2(1 + ah/H)^2} \right].$$

Therefore

$$\boxed{\rho gh = \frac{\varepsilon_0 a E^2}{2(1 + ah/H)^2}},$$

or, equivalently,

$$\boxed{h \left(1 + a \frac{h}{H} \right)^2 = \frac{\varepsilon_0 a E^2}{2\rho g}}.$$

This cubic equation determines the equilibrium height from the supplied quantities. Both b and d cancel, explaining why their values are unnecessary.

Physical explanation. Inserting dielectric increases the capacitance and lowers the electric energy at fixed charge. The corresponding upward force is

$$F_e = - \left. \frac{dU_e}{dh} \right|_Q = \frac{1}{2} \varepsilon_0 a b d \mathcal{E}(h)^2.$$

Its electric pressure is therefore

$$p_e = \frac{F_e}{bd} = \frac{1}{2} \varepsilon_0 a \mathcal{E}(h)^2.$$

The liquid rises until this pressure balances the hydrostatic pressure ρgh associated with the elevated column.

Equating the initial and final total energies would generally describe a possible turning point during motion, rather than the static equilibrium. The equilibrium condition is the vanishing energy derivative used above.

Discussion.

For $a > 0$, define

$$f(h) = h \left(1 + a \frac{h}{H} \right)^2.$$

On $h \geq 0$,

$$f'(h) = \left(1 + a \frac{h}{H} \right) \left(1 + 3a \frac{h}{H} \right) > 0.$$

Thus there is exactly one nonnegative solution. It is stable: increasing h increases the downward gravitational force while decreasing the upward electric force. Explicitly,

$$\frac{d^2(U_e + U_g)}{dh^2} = bd \left[\rho g + \frac{\varepsilon_0 a^2 E^2}{H(1 + ah/H)^3} \right] > 0.$$

The derivation applies to a surface lying between the plate ends. Substituting $h = H$ into the equilibrium equation gives

$$H \varepsilon_r^2 = \frac{\varepsilon_0 (\varepsilon_r - 1) E_c^2}{2\rho g},$$

which yields the threshold E_c stated above. If $E > E_c$, there is no partial-rise equilibrium below the top: the gap becomes completely filled. The detailed surface shape near or above the upper edges requires fringing fields, which the parallel-plate approximation does not determine.

When the capacitance changes only slightly,

$$a \frac{h}{H} \ll 1,$$

the field remains approximately equal to its initial value, and

$$h \simeq \frac{\varepsilon_0(\varepsilon_r - 1)E^2}{2\rho g}.$$

In particular, the rise tends to zero as $E \rightarrow 0$ or $\varepsilon_r \rightarrow 1$.

For an isolated capacitor, increasing the permittivity does not always increase the rise, because it also reduces the field. At fixed E, H , the large-permittivity limit is

$$h \simeq \left(\frac{\varepsilon_0 E^2 H^2}{2\rho g(\varepsilon_r - 1)} \right)^{1/3} \rightarrow 0.$$

More quantitatively, put

$$B = \frac{\varepsilon_0 E^2}{2\rho g H}, \quad x = a \frac{h}{H}.$$

The equilibrium equation gives

$$\frac{h}{H} = \sqrt{B} \frac{\sqrt{x}}{1+x} \leq \frac{\sqrt{B}}{2}.$$

Before imposing the geometrical limit $h \leq H$, the maximum over permittivity is therefore

$$h_{\max} = E \sqrt{\frac{\varepsilon_0 H}{8\rho g}},$$

attained when $x = 1$, or $a = 2/\sqrt{B}$. If this maximum exceeds H , complete filling is possible for a range of permittivities.

Finally, if a voltage source remains connected, $V = Ed$ and $\mathcal{E} = E$ stay constant. Accounting for the source, the effective electric potential energy is $-CV^2/2$. Equilibrium then gives

$$\rho g h = \frac{1}{2} \varepsilon_0 (\varepsilon_r - 1) E^2,$$

leading to the fixed-voltage result in the Answers.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

Problem 4: Plano-convex lens between two media; Billet split lens

A thin lens plane-convex with the diameter $2r$, the curvature radius R and the refractive index n_o is positioned so that on its left side is air ($n_1 = 1$), and on its right side there is a transparent medium with the refractive index $n_2 \neq 1$. The convex face of the lens is directed towards air. In

the air, at the distance s_1 from the lens, measured on the principal optic ax, there is a punctual source of monochromatic light.

a) Demonstrate, using Gauss approximation, that between the position of the image, given by the distance s_2 from the lens, and the position of the light source, exists the relation:

$$\frac{f_1}{s_1} + \frac{f_2}{s_2} = 1$$

where f_1 and f_2 are the focal distances of the lens, in air, respectively in the medium with the refractive index n_2 .

Observation: All the refractive indexes are absolute indexes.

b) The lens is cut perpendicular on its plane face in two equal parts. These parts are moved away at a distance $\delta \ll r$ (Billet lens). On the symmetry axis of the system obtained is led a punctual source of light at the distance s_1 ($s_1 > f_1$) (fig. 4.1). On the right side of the lens there is a screen E at the distance d . The screen is parallel with the plane face of the lens. On this screen there are N interference fringes, if on the right side of the lens is air.

Determine N function of the wave length.

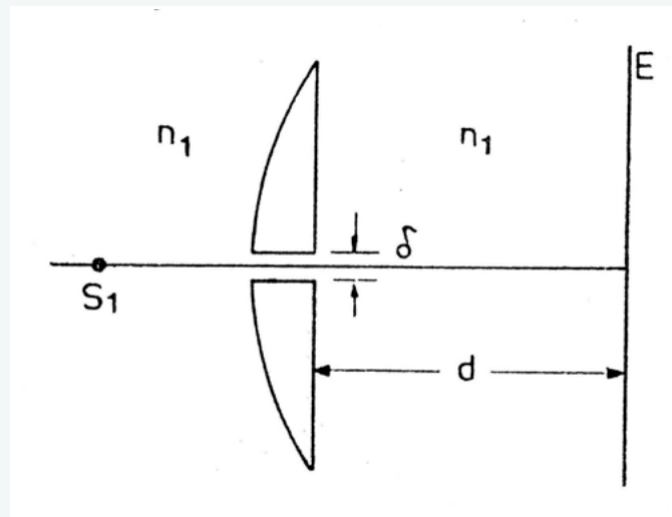


Fig. 4.1

Official English text of the problem.

Answers

- a) The focal distances and image equation are

$$f_1 = \frac{R}{n_o - 1}, \quad f_2 = \frac{n_2 R}{n_o - 1}, \quad \frac{f_1}{s_1} + \frac{f_2}{s_2} = 1.$$

- **b) Put**

$$f = f_1 = \frac{R}{n_o - 1}, \quad v = \frac{fs_1}{s_1 - f}.$$

For the usual screen position, where the outer aperture edges do not limit the interference strip,

$$N \simeq N_0 = \frac{\delta^2(s_1 + d)}{\lambda [d(s_1 - f) - fs_1]}.$$

This requires

$$d > v, \quad r \left(\frac{d}{v} - 1 \right) \geq \delta \left(1 + \frac{d}{s_1} \right).$$

Here N is the usual count obtained by dividing the overlap width by the fringe spacing.

Including finite-aperture clipping, the corresponding count is

$$\bar{N} = \begin{cases} 0, & d \leq v, \\ \max \left\{ 0, \min \left[N_0, \frac{2r\delta}{\lambda f} - N_0 \right] \right\}, & d > v. \end{cases}$$

For positive overlap, the integer number of bright maxima, including boundary maxima, is

$$N_{\text{bright}} = 2 \left\lfloor \frac{\bar{N}}{2} \right\rfloor + 1.$$

Solution

a) Refraction through the two surfaces

Use the thin-lens and Gaussian approximations: neglect the separation of the two surfaces, and retain only first-order terms in ray heights and angles. Take propagation from left to right, with positive s_1 for the real source and positive s_2 for a real image.

Consider a ray incident at height y . Its angle to the optical axis is

$$\theta_1 \simeq \frac{y}{s_1}.$$

Since the centre of curvature is to the right of the convex surface, the surface normal has angle

$$\nu \simeq -\frac{y}{R}.$$

Let θ_o be the ray angle inside the lens. Snell's law in the small-angle approximation gives

$$\theta_1 - \nu = n_o(\theta_o - \nu),$$

hence

$$n_o\theta_o = \theta_1 - \frac{(n_o - 1)y}{R}.$$

At the plane face, whose normal is parallel to the optical axis,

$$n_2\theta_2 = n_o\theta_o.$$

For a ray reaching the axial image,

$$\theta_2 \simeq -\frac{y}{s_2}.$$

Combining these equations and cancelling y yields

$$\boxed{\frac{1}{s_1} + \frac{n_2}{s_2} = \frac{n_o - 1}{R}}.$$

The front focal distance is the source distance that makes the outgoing rays parallel. Setting $s_2 \rightarrow \infty$,

$$f_1 = \frac{R}{n_o - 1}.$$

The rear focal distance is the image distance for parallel incident rays. Setting $s_1 \rightarrow \infty$,

$$f_2 = \frac{n_2 R}{n_o - 1}.$$

Dividing the image equation by $(n_o - 1)/R$ therefore gives

$$\boxed{\frac{f_1}{s_1} + \frac{f_2}{s_2} = 1}.$$

The same relation covers virtual images if s_2 is treated as a signed distance.

b) Images formed by the two halves

Both exterior media are now air, so the focal length is

$$f = \frac{R}{n_o - 1}.$$

Each half forms an image at the longitudinal distance v , where

$$\frac{1}{s_1} + \frac{1}{v} = \frac{1}{f}, \quad v = \frac{f s_1}{s_1 - f} > 0.$$

Choose the transverse coordinate y perpendicular to the cut. The full gap shown in the figure is δ , so the axes of the two halves lie at

$$c_{\pm} = \pm \frac{\delta}{2}.$$

Relative to a half-lens axis, the source height is $-c_{\pm}$. The transverse magnification is $-v/s_1$, giving the image positions

$$y_{\pm} = c_{\pm} + \left(-\frac{v}{s_1}\right)(-c_{\pm}) = \pm \frac{\delta}{2} \left(1 + \frac{v}{s_1}\right).$$

Consequently, the separation of the two images is

$$\boxed{b = \delta \left(1 + \frac{v}{s_1}\right) = \frac{\delta v}{f}}.$$

These images act as coherent secondary sources. Reflection symmetry makes their phases equal, so the central fringe is bright.

Fringe spacing

For a screen beyond the image plane, let

$$L = d - v > 0.$$

At screen coordinate y , the paraxial path difference from the two images is

$$\Delta(y) \simeq \frac{by}{L}.$$

Thus the bright fringes satisfy

$$y_m = \frac{m\lambda L}{b}, \quad m \in \mathbb{Z},$$

and their spacing is

$$\boxed{\beta = \frac{\lambda L}{b} = \frac{\lambda f(d - v)}{\delta v}}.$$

They are straight lines parallel to the cut, within the paraxial approximation.

Width of the overlapping beams

We use geometrical beam envelopes and neglect diffraction at the aperture edges.

The ray through the optical centre of each half is undeviated. These two rays reach the screen at

$$y = \pm H, \quad H = \frac{\delta}{2} \left(1 + \frac{d}{s_1} \right).$$

Provided the outer aperture edges do not clip the overlap, its width is therefore

$$W_0 = 2H = \delta \left(1 + \frac{d}{s_1} \right).$$

Dividing this width by the fringe spacing gives

$$N_0 = \frac{W_0}{\beta} = \frac{\delta^2 v}{\lambda f(d - v)} \left(1 + \frac{d}{s_1} \right).$$

Substituting $v = fs_1/(s_1 - f)$,

$$\boxed{N_0 = \frac{\delta^2(s_1 + d)}{\lambda[d(s_1 - f) - fs_1]}}.$$

For fixed geometry and refractive index, this is proportional to $1/\lambda$.

Finite-aperture conditions

The statement does not locate the screen relative to the image plane, so the overlap conditions must also be checked.

A ray through height y_0 in the upper half passes through its image at height y_+ . Its screen height is

$$y(d) = \left(1 - \frac{d}{v} \right) y_0 + \frac{d}{v} y_+.$$

The upper half occupies

$$\frac{\delta}{2} \leq y_0 \leq \frac{\delta}{2} + r.$$

For $d > v$, define

$$\rho = r \left(\frac{d}{v} - 1 \right).$$

On the central section of the screen, the upper beam occupies

$$H - \rho \leq y \leq H,$$

and the lower beam occupies

$$-H \leq y \leq \rho - H.$$

Their overlap is symmetric about $y = 0$, with width

$$W = \begin{cases} 0, & \rho \leq H, \\ 2(\rho - H), & H < \rho < 2H, \\ 2H, & \rho \geq 2H. \end{cases}$$

Thus the usual expression N_0 applies when

$$\rho \geq 2H \iff r \left(\frac{d}{v} - 1 \right) \geq \delta \left(1 + \frac{d}{s_1} \right).$$

In the partially overlapping case,

$$\frac{W}{\beta} = \frac{2\rho}{\beta} - N_0 = \frac{2r\delta}{\lambda f} - N_0.$$

This proves the general expression for $\overline{N} = W/\beta$ given in the Answers. For $d \leq v$, the two geometrical beams remain on opposite sides of the symmetry axis and do not overlap.

Finally, when $W > 0$, bright maxima occur at the integers satisfying

$$|m| \leq \frac{W}{2\beta}.$$

Including maxima exactly on the boundary gives

$$N_{\text{bright}} = 2 \left\lfloor \frac{W}{2\beta} \right\rfloor + 1.$$

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.