

IPhO 1971

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Problem 1: Two blocks on a triangular prism on a frictionless plane

A triangular prism of mass M is placed one side on a frictionless horizontal plane as shown in Fig. 1. The other two sides are inclined with respect to the plane at angles α_1 and α_2 respectively. Two blocks of masses m_1 and m_2 , connected by an inextensible thread, can slide without friction on the surface of the prism. The mass of the pulley, which supports the thread, is negligible.

- Express the acceleration a of the blocks relative to the prism in terms of the acceleration a_0 of the prism.
- Find the acceleration a_0 of the prism in terms of quantities given and the acceleration g due to gravity.
- At what ratio m_1/m_2 the prism will be in equilibrium?

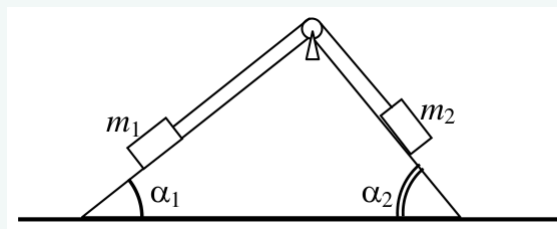


Fig. 1

Official English text of the problem.

Answers

The source has three unnumbered bullets and gives no point values; they are labelled **(1)–(3)** below.

Take a_0 as positive to the right, and a as positive when m_1 accelerates down its face and m_2 accelerates up its face, relative to the prism.

$$1. \quad a = \frac{g(m_1 \sin \alpha_1 - m_2 \sin \alpha_2) + a_0(m_1 \cos \alpha_1 + m_2 \cos \alpha_2)}{m_1 + m_2}$$

$$2. \quad a_0 = \frac{g(m_1 \cos \alpha_1 + m_2 \cos \alpha_2)(m_1 \sin \alpha_1 - m_2 \sin \alpha_2)}{(m_1 + m_2)(M + m_1 + m_2) - (m_1 \cos \alpha_1 + m_2 \cos \alpha_2)^2}$$

$$3. \quad \frac{m_1}{m_2} = \frac{\sin \alpha_2}{\sin \alpha_1}$$

Solution

We use the usual ideal-thread approximation: the thread is massless and taut, and the pulley turns without friction and has negligible inertia. Consequently, the tension T is the same on both sides. The prism translates horizontally, and the blocks remain in contact with their faces.

Because the thread is inextensible and the pulley is fixed to the prism, the blocks have equal signed relative accelerations when measured down the left face and up the right face.

(1) Relative acceleration in terms of a_0

Work in the inertial frame of the horizontal plane. Each block's absolute acceleration equals the prism's acceleration plus its acceleration relative to the prism.

For m_1 , take the positive direction down its incline. The projection of the rightward prism acceleration onto this direction is $-a_0 \cos \alpha_1$. Newton's second law along the incline therefore gives

$$m_1(a - a_0 \cos \alpha_1) = m_1 g \sin \alpha_1 - T.$$

For m_2 , take the positive direction up its incline. The corresponding equation is

$$m_2(a - a_0 \cos \alpha_2) = T - m_2 g \sin \alpha_2.$$

The normal forces have no components along the inclines. Adding the equations eliminates T :

$$(m_1 + m_2)a = g(m_1 \sin \alpha_1 - m_2 \sin \alpha_2) + a_0(m_1 \cos \alpha_1 + m_2 \cos \alpha_2).$$

Dividing by $m_1 + m_2$ gives answer (1).

(2) Acceleration of the prism

There is no external horizontal force on the complete system: gravity and the supporting plane's normal force are vertical. Hence its horizontal momentum is conserved, so its centre of mass has zero horizontal acceleration.

For positive a , both blocks accelerate leftward relative to the prism. Their absolute horizontal accelerations are

$$a_{1x} = a_0 - a \cos \alpha_1, \quad a_{2x} = a_0 - a \cos \alpha_2.$$

Thus

$$Ma_0 + m_1(a_0 - a \cos \alpha_1) + m_2(a_0 - a \cos \alpha_2) = 0,$$

or

$$(M + m_1 + m_2)a_0 = (m_1 \cos \alpha_1 + m_2 \cos \alpha_2)a.$$

For compactness, define

$$S = m_1 + m_2, \quad C = m_1 \cos \alpha_1 + m_2 \cos \alpha_2, \quad D = m_1 \sin \alpha_1 - m_2 \sin \alpha_2.$$

The two equations become

$$Sa = gD + Ca_0, \quad (M + S)a_0 = Ca.$$

Eliminating a ,

$$[S(M + S) - C^2]a_0 = gCD,$$

which gives answer (2).

For the inclined faces shown, $0 < C < S$, so

$$S(M + S) - C^2 = MS + (S^2 - C^2) > 0.$$

Therefore the prism accelerates rightward if $m_1 \sin \alpha_1 > m_2 \sin \alpha_2$, and leftward if the inequality is reversed.

(3) Condition for equilibrium of the prism

Equilibrium requires $a_0 = 0$. Since $C > 0$, horizontal momentum conservation then also requires $a = 0$. The equations along the inclines reduce to

$$T = m_1 g \sin \alpha_1 = m_2 g \sin \alpha_2.$$

Consequently,

$$\boxed{\frac{m_1}{m_2} = \frac{\sin \alpha_2}{\sin \alpha_1}}.$$

At this ratio, a prism initially at rest remains at rest.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

Problem 2: Hydrogen in a tube closed by a mercury column

A vertical glass tube of cross section $S = 1.0 \text{ cm}^2$ contains unknown amount of hydrogen. The upper end of the tube is closed. The other end is opened and is immersed in a pan filled with mercury. The tube and the pan are placed in a sealed chamber containing air at temperature $T_0 = 273 \text{ K}$ and pressure $P_0 = 1.334 \times 10^5 \text{ Pa}$. Under these conditions the height of mercury column in the tube above the mercury level in the pan is $h_0 = 0.70 \text{ m}$.

One of the walls of the chamber is a piston, which expands the air isothermally to a pressure of $P_1 = 8.00 \times 10^4 \text{ Pa}$. As a result the height of the mercury column in the tube decreases to $h_1 = 0.40 \text{ m}$. Then the chamber is heated up at a constant volume to some temperature T_2 until the mercury column rises to $h_2 = 0.50 \text{ m}$. Finally, the air in the chamber is expanded at constant pressure and the mercury level in the tube settles at $h_3 = 0.45 \text{ m}$ above the mercury level in the pan.

Provided that the system is in mechanical and thermal equilibrium during all the processes calculate the mass m of the hydrogen, the intermediate temperature T_2 , and the pressure P in the final state.

The density of mercury at temperature T_0 is $\rho_0 = 1.36 \times 10^4 \text{ kg/m}^3$, the coefficient of expansion for mercury $\beta = 1.84 \times 10^{-4} \text{ K}^{-1}$, and the gas constant $R = 8.314 \text{ J/(mol} \times \text{K)}$. The thermal expansion of the glass tube and the variations of the mercury level in the pan are not considered.

Hint. If ΔT is the interval of temperature variations of the system then $\beta \Delta T = x \ll 1$. In that case you can use the approximation:

$$\frac{1}{1+x} \approx 1-x.$$

Official English text of the problem.

Answers

Question 2 has no numbered subparts. In the requested order:

- **Hydrogen mass:** $m \approx 2.1 \times 10^{-6} \text{ kg} = 2.1 \text{ mg}$.
- **Intermediate temperature:** $T_2 \approx 358 \text{ K}$.
- **Final hydrogen pressure:** $P_{\text{H}_2} \approx 4.7 \times 10^4 \text{ Pa}$. The final chamber-air pressure is $P_{\text{air}} \approx 1.05 \times 10^5 \text{ Pa}$; the statement does not say whether "the pressure P in the final state" means the hydrogen pressure or the chamber-air pressure, so both are given.

Solution

Treat hydrogen and air as ideal gases, with negligible hydrostatic pressure variations within the gases. Use the standard values

$$g = 9.81 \text{ m s}^{-2}, \quad M_{\text{H}_2} = 2.0 \times 10^{-3} \text{ kg mol}^{-1}.$$

Let H be the height of the tube's closed end above the mercury surface in the pan. Since glass expansion and changes of the pan level are neglected, H is constant. At state i , the hydrogen volume is

$$V_i = S(H - h_i).$$

Write P_i for the chamber-air pressure and p_i for the hydrogen pressure. Hydrostatic equilibrium in the connected mercury gives

$$p_i = P_i - \rho_i g h_i.$$

Thermal equilibrium means that both gases and the mercury have the same temperature at each state.

Hydrogen mass m .

The first expansion is isothermal at T_0 , so the mercury density remains ρ_0 . Thus

$$\begin{aligned} p_0 &= P_0 - \rho_0 g h_0 = 4.00088 \times 10^4 \text{ Pa}, \\ p_1 &= P_1 - \rho_0 g h_1 = 2.66336 \times 10^4 \text{ Pa}. \end{aligned}$$

The fixed amount of hydrogen obeys Boyle's law:

$$p_0 S(H - h_0) = p_1 S(H - h_1).$$

Consequently,

$$H = \frac{p_0 h_0 - p_1 h_1}{p_0 - p_1} \approx 1.29738 \text{ m}.$$

Using $S = 1.0 \times 10^{-4} \text{ m}^2$, the amount of hydrogen is

$$n = \frac{p_0 S(H - h_0)}{RT_0} \approx 1.0530 \times 10^{-3} \text{ mol}.$$

Therefore,

$$m = n M_{\text{H}_2} \approx 2.1 \times 10^{-6} \text{ kg} = 2.1 \text{ mg}.$$

For the remaining calculations, define the constant

$$C = \frac{nR}{S} \approx 87.5475 \text{ Pa m K}^{-1}.$$

The hydrogen ideal-gas equation then becomes

$$p_i = \frac{CT_i}{H - h_i}.$$

Intermediate temperature T_2 .

Mercury's volume expansion gives its temperature-dependent density:

$$\rho(T) = \frac{\rho_0}{1 + \beta(T - T_0)} \approx \rho_0 [1 - \beta(T - T_0)],$$

where the final expression uses the supplied first-order approximation.

During the constant-volume heating, the chamber air satisfies

$$P_2 = P_1 \frac{T_2}{T_0}.$$

Combining this with hydrostatic equilibrium and the hydrogen gas law at h_2 ,

$$\frac{CT_2}{H - h_2} = P_1 \frac{T_2}{T_0} - \rho_0 g h_2 [1 - \beta(T_2 - T_0)].$$

Collecting the terms containing T_2 gives

$$\left[\frac{P_1}{T_0} - \frac{C}{H - h_2} + \beta \rho_0 g h_2 \right] T_2 = \rho_0 g h_2 (1 + \beta T_0).$$

Hence

$$T_2 = \frac{\rho_0 g h_2 (1 + \beta T_0)}{P_1/T_0 - C/(H - h_2) + \beta \rho_0 g h_2} \approx 358.32 \text{ K},$$

so

$$\boxed{T_2 \approx 358 \text{ K.}}$$

The chamber-air pressure at this stage is

$$P_2 = P_1 \frac{T_2}{T_0} \approx 1.05002 \times 10^5 \text{ Pa.}$$

Pressure in the final state.

The final expansion occurs at constant chamber-air pressure. Therefore,

$$\boxed{P_3 = P_2 \approx 1.05 \times 10^5 \text{ Pa.}}$$

To determine the hydrogen pressure, its final temperature T_3 must also be found. At $h_3 = 0.45 \text{ m}$,

$$p_3 = \frac{CT_3}{H - h_3} = P_2 - \rho_0 g h_3 [1 - \beta(T_3 - T_0)].$$

Rearranging,

$$\left[\frac{C}{H - h_3} - \beta \rho_0 g h_3 \right] T_3 = P_2 - \rho_0 g h_3 (1 + \beta T_0).$$

Thus

$$T_3 = \frac{P_2 - \rho_0 g h_3 (1 + \beta T_0)}{C/(H - h_3) - \beta \rho_0 g h_3} \approx 454.64 \text{ K.}$$

The final hydrogen pressure is consequently

$$p_3 = \frac{CT_3}{H - h_3} \approx 4.6971 \times 10^4 \text{ Pa,}$$

or

$$P_{\text{H}_2} \approx 4.7 \times 10^4 \text{ Pa.}$$

Finally, the expansion parameters are

$$\beta(T_2 - T_0) \approx 0.0157, \quad \beta(T_3 - T_0) \approx 0.0334.$$

Both are small, consistent with the approximation requested in the hint.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

Problem 3: Cube-shaped circuit of batteries, capacitors and resistors

Four batteries of EMF $E_1 = 4 \text{ V}$, $E_2 = 8 \text{ V}$, $E_3 = 12 \text{ V}$, and $E_4 = 16 \text{ V}$, four capacitors with the same capacitance $C_1 = C_2 = C_3 = C_4 = 1 \mu\text{F}$, and four equivalent resistors are connected in the circuit shown in Fig. 3. The internal resistance of the batteries is negligible.

- Calculate the total energy W accumulated on the capacitors when a steady state of the system is established.
- The points H and B are short connected. Find the charge on the capacitor C_2 in the new steady state.

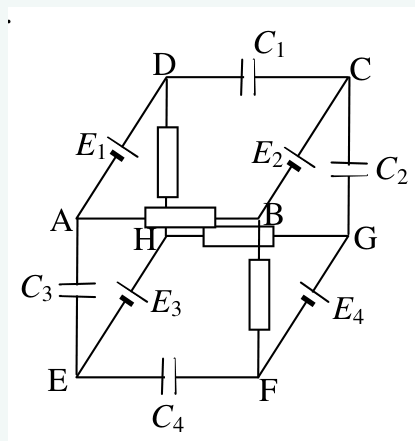


Fig. 3

Official English text of the problem.

Answers

The two unnumbered bullets are designated (1) and (2) below.

1. Total energy stored: $W = 26 \mu\text{J}$.

2. Charge on capacitor C_2 after shorting H and B: $Q_2 = 0 \text{ C}$.

Solution

Let each resistor have resistance R . The resistors occupy edges AB , BF , DH , and HG ; crossings in the projection do not create additional junctions.

The long battery plates indicate the positive terminals, so

$$\begin{aligned} V_D - V_A &= E_1, & V_C - V_B &= E_2, \\ V_H - V_E &= E_3, & V_G - V_F &= E_4. \end{aligned}$$

In a steady state, capacitor voltages are constant and no current flows through the capacitors.

(1) Total energy stored

With the capacitor branches open, the only conducting loop is

$$F \longrightarrow G \longrightarrow H \longrightarrow D \longrightarrow A \longrightarrow B \longrightarrow F.$$

The branches containing E_2 and E_3 carry no steady current.

Kirchhoff's voltage law around the conducting loop gives

$$E_4 - E_1 - 4IR = 0.$$

Thus the voltage drop across each resistor is

$$IR = \frac{E_4 - E_1}{4} = \frac{16 - 4}{4} \text{ V} = 3 \text{ V}.$$

Taking $V_F = 0$, the battery rise and successive voltage drops give

$$\begin{aligned} V_G &= 16 \text{ V}, & V_H &= 13 \text{ V}, & V_D &= 10 \text{ V}, \\ V_A &= 6 \text{ V}, & V_B &= 3 \text{ V}. \end{aligned}$$

The remaining batteries fix

$$V_C = V_B + E_2 = 11 \text{ V}, \quad V_E = V_H - E_3 = 1 \text{ V}.$$

Consequently, the capacitor voltages are:

Capacitor	Voltage magnitude	Positive plate
C_1	$ V_D - V_C = 1 \text{ V}$	C
C_2	$ V_C - V_G = 5 \text{ V}$	G
C_3	$ V_A - V_E = 5 \text{ V}$	A
C_4	$ V_E - V_F = 1 \text{ V}$	E

Using the electrostatic energy $W_i = \frac{1}{2} C_i U_i^2$,

$$W = \frac{1}{2} (1 \times 10^{-6} \text{ F}) (1^2 + 5^2 + 5^2 + 1^2) \text{ V}^2.$$

Therefore,

$$W = 2.6 \times 10^{-5} \text{ J} = 26 \mu\text{J}.$$

(2) Charge on C_2 after shorting H and B

The short circuit makes $V_H = V_B$. Choose their common potential as zero:

$$V_H = V_B = 0.$$

Battery E_2 then fixes

$$V_C = E_2 = 8 \text{ V}.$$

In the new steady state, the capacitors again carry no current. Battery E_4 drives a loop through the two equal resistors GH and BF , so

$$I_4 = \frac{E_4}{2R}.$$

The potential of G relative to H is therefore

$$V_G - V_H = I_4 R = \frac{E_4}{2} = 8 \text{ V}.$$

Hence $V_G = V_C$, and the voltage across C_2 vanishes:

$$U_2 = V_C - V_G = E_2 - \frac{E_4}{2} = 0.$$

Its final charge is consequently

$$Q_2 = C_2 |U_2| = 0 \text{ C}.$$

The solution was checked against the official solution.

Problem 4: Fish in a spherical aquarium in front of a plane mirror

A spherical aquarium, filled with water, is placed in front of a flat vertical mirror. The radius of the aquarium is R , and the distance between its center and the mirror is $3R$. A small fish, which is initially at the point nearest to the mirror, starts to move with velocity v along the wall. An observer looks at the fish from a very large distance along a horizontal line passing through the center of the aquarium.

What is the relative velocity v_{rel} at which the two images of the fish seen by the observer will move apart? Express your answer in terms of v . Assume that:

- The wall of the aquarium is made of a very thin glass.
- The index of refraction of water is $n = 4/3$.

Official English text of the problem.

Answers

- 4. The initial relative velocity is

$$v_{\text{rel}} = \frac{8}{3}v.$$

The direct image moves at $2v$, while the image seen through reflection moves in the opposite direction at $2v/3$.

Solution

4. Take the viewing axis as the x -axis, directed from the observer toward the mirror. Put the aquarium centre at $x = 0$, the mirror at $x = 3R$, and the fish initially at $x = R$.

Choose the transverse y -direction along the fish's initial motion. Because the fish moves tangentially to the spherical wall, its displacement to first order in time is

$$y = vt,$$

while its axial displacement is of second order. Thus we may keep its axial coordinate equal to R when calculating the **initial** image velocities.

The thin glass can be neglected: refractions at its two nearly coincident surfaces combine into refraction directly between water and air. We use paraxial Snell's law, which is sufficient for the initial motion about the axis.

By reversibility of light, trace rays backward from the distant observer toward the fish. To the required accuracy, these rays initially travel parallel to the axis. Their height h at the observer-facing surface, $x = -R$, gives the apparent transverse image position.

At this surface, the normal directed into the water makes the small angle $-h/R$ with the axis. If the refracted ray makes angle α , Snell's law gives

$$0 + \frac{h}{R} = n \left(\alpha + \frac{h}{R} \right).$$

Consequently,

$$\alpha = -\frac{n-1}{n} \frac{h}{R} = -\frac{h}{4R}.$$

After crossing the aquarium, its height at the mirror-facing surface, $x = R$, is

$$k = h + 2R\alpha = \frac{h}{2}.$$

Direct image. For the direct path, the backward-traced ray reaches the fish at this surface, so $k = y$. Therefore

$$h_d = 2y, \quad \boxed{\dot{h}_d = 2v}.$$

Image seen through reflection. For this path, the backward-traced ray continues through the mirror-facing surface toward the mirror. The outward surface normal makes angle k/R with the axis. If the ray's angle in air is γ , Snell's law gives

$$n \left(\alpha - \frac{k}{R} \right) = \gamma - \frac{k}{R}.$$

Using $n = 4/3$, $\alpha = -h/(4R)$, and $k = h/2$,

$$\gamma = n\alpha - (n-1)\frac{k}{R} = -\frac{h}{2R}.$$

We can unfold the plane-mirror reflection by replacing the fish with its mirror image. Its axial coordinate is

$$x_{\text{mirror image}} = 2(3R) - R = 5R,$$

and its transverse coordinate remains y . Since the fish is at the wall, refraction where the ray finally reaches it introduces no additional propagation distance.

The unfolded ray travels an axial distance $4R$ from the aquarium's mirror-facing surface to this mirrored position. Hence

$$y = k + 4R\gamma = \frac{h}{2} + 4R \left(-\frac{h}{2R} \right) = -\frac{3h}{2}.$$

Thus the reflected image has apparent transverse position and velocity

$$h_r = -\frac{2}{3}y, \quad \boxed{\dot{h}_r = -\frac{2}{3}v}.$$

The two images move in opposite transverse directions. Their initial relative velocity is therefore

$$\boxed{v_{\text{rel}} = |\dot{h}_d - \dot{h}_r| = 2v + \frac{2}{3}v = \frac{8}{3}v}.$$

The solution was checked against the official solution.