

IPhO 1970

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Problem 1: Motor-driven carriage on a bar sliding on a table

A long bar with the mass $M = 1$ kg is placed on a smooth horizontal surface of a table where it can move frictionless. A carriage equipped with a motor can slide along the upper horizontal panel of the bar, the mass of the carriage is $m = 0.1$ kg. The friction coefficient of the carriage is $\mu = 0.02$.

The motor is winding a thread around a shaft at a constant speed $v_0 = 0.1$ m/s. The other end of the thread is tied up to a rather distant stationary support in one case (Fig.1, a), whereas in the other case it is attached to a picket at the edge of the bar (Fig.1, b). While holding the bar fixed one allows the carriage to start moving at the velocity V_0 then the bar is let loose.

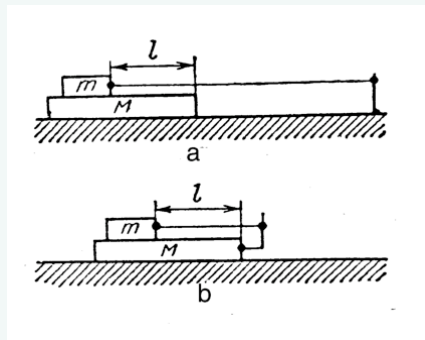


Fig. 1

By the moment the bar is released the front edge of the carriage is at the distance $l = 0.5$ m from the front edge of the bar. For both cases find the laws of movement of both the bar and the carriage and the time during which the carriage will reach the front edge of the bar.

Official English text of the problem.

Answers

Take $t = 0$ at release and the forward direction as positive. Let x_b and x_c denote the bar's and carriage's displacements from their respective initial positions. Using $g = 9.8 \text{ m s}^{-2}$, define

$$a = \frac{\mu mg}{M} = 0.0196 \text{ m s}^{-2}, \quad t_* = \frac{v_0}{a} \approx 5.1 \text{ s}.$$

- (a) Thread attached to the stationary support:

$$x_c(t) = v_0 t,$$

$$x_b(t) = \begin{cases} \frac{1}{2} a t^2, & 0 \leq t \leq t_*, \\ v_0 t - \frac{v_0^2}{2a}, & t \geq t_*. \end{cases}$$

After t_* , both bodies move at 0.1 m s^{-1} . **The carriage never reaches the front edge:** the remaining gap is

$$d_{\min} = l - \frac{M v_0^2}{2 \mu m g} \approx 0.25 \text{ m}.$$

- (b) Thread attached to the bar:

$$x_b(t) = 0, \quad x_c(t) = v_0 t.$$

The bar remains at rest, and the carriage reaches its front edge after

$$t_{\text{edge}} = \frac{l}{v_0} = 5.0 \text{ s}.$$

Solution

Treat the thread as massless and inextensible, and interpret v_0 as its constant linear winding speed. The thread is horizontal, so the normal force on the carriage is mg . During sliding, the friction force has magnitude

$$f = \mu m g.$$

The table exerts no horizontal force on the bar.

Before release, the bar and the thread's attachment point are stationary in both arrangements. Consequently, the carriage's initial velocity is

$$V_0 = v_0.$$

Removing the restraint produces no impulse, so immediately after release

$$v_b(0) = 0, \quad v_c(0) = v_0.$$

(a) Thread attached to the stationary support

Because the thread's far end is fixed, winding it at speed v_0 makes the carriage approach that end at the same speed:

$$v_c = v_0, \quad x_c = v_0 t.$$

Initially the carriage slides forward relative to the bar. Friction acts backward on the carriage and forward on the bar. If T is the thread tension, Newton's second law gives

$$m \dot{v}_c = T - f = 0, \quad M \dot{v}_b = f.$$

Thus $T = f$, and the bar has constant acceleration

$$a = \frac{f}{M} = \frac{\mu m g}{M}.$$

As long as the carriage is sliding forward,

$$v_b = at, \quad x_b = \frac{1}{2}at^2.$$

The relative velocity is

$$v_c - v_b = v_0 - at.$$

It vanishes at

$$t_* = \frac{v_0}{a} = \frac{Mv_0}{\mu mg} \approx 5.1 \text{ s}.$$

The carriage's forward displacement relative to the bar up to this instant is

$$s_{\max} = x_c(t_*) - x_b(t_*) = v_0 t_* - \frac{1}{2}at_*^2 = \frac{v_0^2}{2a} = \frac{Mv_0^2}{2\mu mg}.$$

Numerically,

$$s_{\max} = \frac{1(0.1)^2}{2(0.02)(0.1)(9.8)} \approx 0.255 \text{ m}.$$

Since $s_{\max} < l = 0.5 \text{ m}$, the carriage has not reached the edge when the relative motion stops.

At t_* , both bodies have velocity v_0 . They can thereafter continue together without horizontal forces: friction and tension are both zero. The sliding-friction acceleration must therefore cease at this instant. For $t \geq t_*$,

$$v_b = v_c = v_0,$$

and

$$x_b(t) = x_b(t_*) + v_0(t - t_*) = v_0 t - \frac{v_0^2}{2a}.$$

The separation between the front edges is

$$d(t) = l + x_b(t) - x_c(t) = \begin{cases} l - v_0 t + \frac{1}{2}at^2, & 0 \leq t \leq t_*, \\ l - \frac{v_0^2}{2a}, & t \geq t_*. \end{cases}$$

Its final value is approximately 0.245 m, so **there is no finite arrival time**.

(b) Thread attached to the bar

Here the motor shortens the distance between the carriage and a point fixed to the bar. Therefore,

$$v_c - v_b = v_0.$$

All horizontal forces are internal to the bar–carriage system: the thread pulls the carriage forward and the bar backward, while friction supplies an equal-and-opposite pair in the other directions. Hence total horizontal momentum is conserved:

$$Mv_b + mv_c = mv_0,$$

where the right-hand side is the momentum at release.

Substituting $v_c = v_b + v_0$,

$$Mv_b + m(v_b + v_0) = mv_0,$$

so

$$v_b = 0, \quad v_c = v_0.$$

Thus

$$x_b(t) = 0, \quad x_c(t) = v_0 t.$$

The tension equals the sliding friction, $T = \mu mg$, giving zero net force on each body.

The front-edge separation decreases uniformly:

$$d(t) = l - v_0 t.$$

Setting $d = 0$ gives

$$t_{\text{edge}} = \frac{0.5 \text{ m}}{0.1 \text{ m s}^{-1}} = 5.0 \text{ s}.$$

The solution was checked against the official solution.

Problem 2: Mass of the hydrogen atom from the NaCl unit cell

A unit cell of a crystal of sodium chloride (common salt- NaCl) is a cube with the edge length $a = 5.6 \cdot 10^{-10} \text{ m}$ (Fig.2). The black circles in the figure stand for the position of sodium atoms whereas the white ones are chlorine atoms. The entire crystal of common salt turns out to be a repetition of such unit cells. The relative atomic mass of sodium is 23 and that of chlorine is 35.5. The density of the common salt $\rho = 2.22 \cdot 10^3 \text{ kg/m}^3$. Find the mass of a hydrogen atom.

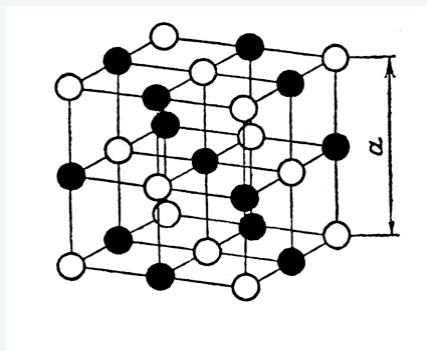


Fig. 2

Official English text of the problem.

Answers

- **Problem 2.** $m_{\text{H}} \approx 1.67 \times 10^{-27} \text{ kg}$, or $1.7 \times 10^{-27} \text{ kg}$ to two significant figures.

Solution

Problem 2.

The mass assigned to one unit cell is its volume multiplied by the crystal density:

$$M_{\text{cell}} = \rho a^3.$$

To count the atoms belonging to a cell, we account for sharing between neighbouring cells. An atom at a corner belongs to eight cells, one at an edge midpoint to four, and one at a face centre to two. An atom inside the cube belongs entirely to that cell.

The white circles represent chlorine atoms at the eight corners and six face centres. Their effective number per cell is

$$N_{\text{Cl}} = 8 \left(\frac{1}{8} \right) + 6 \left(\frac{1}{2} \right) = 4.$$

The black circles represent sodium atoms at the twelve edge midpoints and the body centre:

$$N_{\text{Na}} = 12 \left(\frac{1}{4} \right) + 1 = 4.$$

Thus each unit cell contains four NaCl formula units.

At the accuracy intended here, take hydrogen's relative atomic mass as 1 and neglect mass changes due to chemical binding. The given relative atomic masses then imply

$$m_{\text{Na}} \simeq 23m_{\text{H}}, \quad m_{\text{Cl}} \simeq 35.5m_{\text{H}}.$$

Consequently,

$$\rho a^3 = 4m_{\text{Na}} + 4m_{\text{Cl}} \simeq 4(23 + 35.5)m_{\text{H}} = 234m_{\text{H}}.$$

Solving for the hydrogen-atom mass gives

$$m_{\text{H}} \simeq \frac{\rho a^3}{234} = \frac{(2.22 \times 10^3 \text{ kg/m}^3)(5.6 \times 10^{-10} \text{ m})^3}{234} = 1.666 \dots \times 10^{-27} \text{ kg}.$$

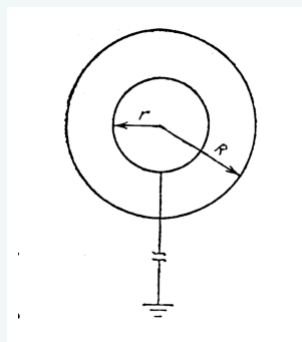
Since the edge length is given to two significant figures,

$$\boxed{m_{\text{H}} \simeq 1.7 \times 10^{-27} \text{ kg}}.$$

The solution was checked against the official solution.

Problem 3: Earthed metal ball inside a charged spherical shell

Inside a thin-walled metal sphere with radius $R = 20 \text{ cm}$ there is a metal ball with the radius $r = 10 \text{ cm}$ which has a common centre with the sphere. The ball is connected with a very long wire to the Earth via an opening in the sphere (Fig. 3). A charge $Q = 10^{-8} \text{ C}$ is placed onto the outside sphere. Calculate the potential of this sphere, electrical capacity of the obtained system of conducting bodies and draw out an equivalent electric scheme.



Answers

Problem 3 has no numbered subparts. In the requested order:

- **Potential of the outer sphere relative to Earth:**

$$V = \frac{Q(R-r)}{4\pi\epsilon_0 R^2} \approx 225 \text{ V.}$$

- **Electrical capacity of the system:**

$$C = \frac{4\pi\epsilon_0 R^2}{R-r} \approx 4.45 \times 10^{-11} \text{ F} = 44.5 \text{ pF.}$$

- **Equivalent electric scheme:** two capacitors connected **in parallel** between the outer sphere and Earth:

$$C_{\text{gap}} = \frac{4\pi\epsilon_0 rR}{R-r} \approx 22.3 \text{ pF}, \quad C_{\text{ext}} = 4\pi\epsilon_0 R \approx 22.3 \text{ pF.}$$

Solution

We neglect the disturbance caused by the small opening and the connecting wire, and treat Earth as a distant conductor at zero potential. The shell thickness is negligible, so both its surfaces have radius R .

The outer shell is electrically isolated and retains total charge Q . The inner ball, however, can exchange charge with Earth.

Potential of the outer sphere.

Let the final charge on the inner ball be q . In electrostatic equilibrium, the electric field inside the metal of the shell vanishes. A Gaussian surface within that metal therefore encloses zero net charge.

Consequently, the shell's inner surface carries charge $-q$, and its outer surface carries

$$Q_{\text{outer}} = Q + q.$$

Write $k = 1/(4\pi\epsilon_0)$. A uniformly charged spherical surface produces constant potential kQ/R everywhere inside it. Thus the potential of the inner ball is

$$V_{\text{ball}} = k \left(\frac{q}{r} + \frac{Q}{R} \right).$$

Because the ball is grounded,

$$V_{\text{ball}} = 0 \implies q = -Q \frac{r}{R}.$$

Numerically,

$$q = -5.0 \times 10^{-9} \text{ C.}$$

The inner and outer surfaces of the shell therefore each carry $+5.0 \times 10^{-9} \text{ C}$.

At the shell, the potential is

$$V = k \frac{Q + q}{R} = k \frac{Q}{R} \left(1 - \frac{r}{R}\right) = \frac{Q(R - r)}{4\pi\epsilon_0 R^2}.$$

Using $k \simeq 8.99 \times 10^9 \text{ N m}^2 \text{ C}^{-2}$,

$$V = (8.99 \times 10^9)(10^{-8}) \frac{0.20 - 0.10}{(0.20)^2} \approx \boxed{225 \text{ V}}.$$

Electrical capacity of the system.

The capacitance is the charge placed on the isolated shell divided by its potential relative to Earth:

$$C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{R^2}{R - r}.$$

Hence

$$C = \frac{1}{8.99 \times 10^9} \frac{(0.20)^2}{0.20 - 0.10} \approx \boxed{4.45 \times 10^{-11} \text{ F} = 44.5 \text{ pF}}.$$

Equivalent electric scheme.

There are two contributions to the charge stored on the shell.

The field in the gap corresponds to a spherical capacitor between the shell and the grounded ball. Its potential difference is

$$V = k(-q) \left(\frac{1}{r} - \frac{1}{R} \right),$$

so its capacitance is

$$C_{\text{gap}} = \frac{-q}{V} = \frac{4\pi\epsilon_0 r R}{R - r}.$$

The exterior field corresponds to a capacitor between the shell's outer surface and distant Earth. Since

$$V = k \frac{Q_{\text{outer}}}{R},$$

this contribution has capacitance

$$C_{\text{ext}} = \frac{Q_{\text{outer}}}{V} = 4\pi\epsilon_0 R.$$

Both capacitors share the shell as one terminal. Their other terminals—the ball and Earth—are connected by the grounding wire, so they are in parallel:

$$C = C_{\text{gap}} + C_{\text{ext}} = 4\pi\epsilon_0 \left(\frac{rR}{R - r} + R \right) = \frac{4\pi\epsilon_0 R^2}{R - r}.$$

To draw the scheme, mark a node S for the outer shell and split the connection into two branches. Put one capacitor in each branch, labelled $C_{\text{gap}} = 22.3 \text{ pF}$ and $C_{\text{ext}} = 22.3 \text{ pF}$. Join their other ends at a common node with an Earth symbol. The potential difference across each capacitor is 225 V , with the shell terminal positive.

The solution was checked against the official solution.

Problem 4: Spherical telescope mirror and a disk receiver in its focus

A spherical mirror is installed into a telescope. Its lateral diameter is $D = 0,5 \text{ m}$ and the radius of the curvature $R = 2 \text{ m}$. In the main focus of the mirror there is an emission receiver in the form of a round disk. The disk is placed perpendicular to the optical axis of the mirror (Fig. 4). What should the radius r of the receiver be so that it could receive the entire flux of the emission reflected by the mirror? How would the received flux of the emission decrease if the detector's dimensions decreased by 8 times?

Directions: 1) When calculating small values α ($\alpha \ll 1$) one may perform a substitution $\sqrt{1 - \alpha} \approx 1 - \frac{\alpha}{2}$; 2) diffraction should not be taken into account.

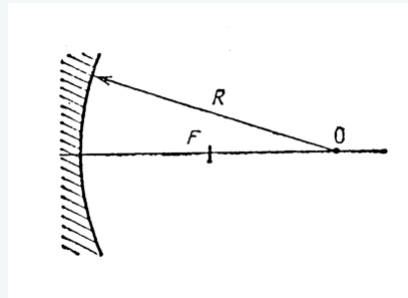


Fig. 4

Official English text of the problem.

Answers

- **Required receiver radius:**

$$r_{\min} \simeq \frac{D^3}{16R^2} \approx 2.0 \text{ mm.}$$

Without the small-aperture expansion, the geometrical result is $2.024 \dots \text{ mm}$.

- **Flux after an eightfold reduction in linear dimensions:**

$$\frac{\Phi_{\text{small}}}{\Phi_{\text{full}}} \simeq \frac{1}{4}.$$

The received flux decreases by approximately a factor of **4**, corresponding to a **75% loss**. Unexpanded ray geometry gives a retained fraction of $0.25449 \dots$

Solution

Required receiver radius

Assume a distant source on the optical axis, providing parallel rays and uniform illumination across the mirror's aperture. We use geometrical optics and neglect diffraction.

Let the mirror's vertex be at $z = 0$, with its centre of curvature at $z = R$. Its paraxial focus is therefore at

$$f = \frac{R}{2} = 1 \text{ m.}$$

Write the aperture radius as

$$a = \frac{D}{2} = 0.25 \text{ m.}$$

Consider an incident ray at height h above the axis. At its point of reflection P , let the normal make an angle θ with the axis. The spherical surface gives

$$h = R \sin \theta, \quad z_P = R(1 - \cos \theta).$$

By the law of reflection, the reflected ray travels towards the axis at angle 2θ . Its signed height when it reaches the focal plane $z = f$ is

$$y_F = h - (f - z_P) \tan 2\theta.$$

Substituting the expressions above,

$$\begin{aligned} y_F &= R \sin \theta - R \left(\cos \theta - \frac{1}{2} \right) \tan 2\theta \\ &= -\frac{R \sin \theta (1 - \cos \theta)}{\cos 2\theta}. \end{aligned}$$

The negative sign means that the ray has already crossed the axis before reaching the paraxial focal plane: this is spherical aberration.

Thus its distance from the axis in that plane is

$$\boxed{\rho(h) = \frac{h \left(1 - \sqrt{1 - h^2/R^2} \right)}{1 - 2h^2/R^2}.$$

For the present aperture, the numerator increases with h , while the positive denominator decreases. Hence $\rho(h)$ increases, and the outermost rays determine the required disk radius:

$$r_{\min} = \rho(a) = \frac{a \left(1 - \sqrt{1 - a^2/R^2} \right)}{1 - 2a^2/R^2}.$$

With the given values, this is

$$r_{\min} = 2.024 \dots \times 10^{-3} \text{ m.}$$

To obtain the requested small-quantity approximation, note that

$$\frac{h^2}{R^2} \leq \frac{a^2}{R^2} = \frac{1}{64} \ll 1.$$

Using the supplied expansion,

$$1 - \sqrt{1 - \frac{h^2}{R^2}} \simeq \frac{h^2}{2R^2}.$$

Keeping the leading nonzero term in $\rho(h)$ therefore gives

$$\rho(h) \simeq \frac{h^3}{2R^2}.$$

Consequently,

$$r_{\min} \simeq \frac{a^3}{2R^2} = \frac{D^3}{16R^2} = \frac{(0.5)^3}{16(2)^2} \text{ m} = 1.95 \times 10^{-3} \text{ m} \approx 2.0 \text{ mm}.$$

Flux after reducing the detector's dimensions

An eightfold reduction in linear dimensions makes the detector radius

$$r_{\text{small}} = \frac{r_{\min}}{8}.$$

Since $\rho(h)$ increases with h , this smaller disk receives the rays reflected from a central portion of the mirror's aperture, of radius b , defined by

$$\rho(b) = \frac{\rho(a)}{8}.$$

Using the cubic dependence just derived,

$$\frac{b^3}{2R^2} = \frac{1}{8} \frac{a^3}{2R^2},$$

so

$$b = \frac{a}{2}.$$

For uniform incident illumination and uniform reflectivity, the reflected flux is proportional to the **projected aperture area** contributing to the detector. Therefore,

$$\frac{\Phi_{\text{small}}}{\Phi_{\text{full}}} = \frac{\pi b^2}{\pi a^2} \simeq \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Thus the received flux decreases approximately fourfold.

For comparison, using the unexpanded expression for $\rho(h)$, the equation $\rho(b) = \rho(a)/8$ gives $b = 0.126117 \dots \text{ m}$, and hence

$$\frac{\Phi_{\text{small}}}{\Phi_{\text{full}}} = 0.25449 \dots,$$

consistent with the leading-order result.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.