

IPhO 1969

Theory problems and solutions · Brno, Czechoslovakia · 4 theory problems with complete solutions

eulersolve.org/ipro/1969/

Problem 1: Three carts connected by a string over a pulley

Figure 1 shows a mechanical system consisting of three carts A , B and C of masses $m_1 = 0.3 \text{ kg}$, $m_2 = 0.2 \text{ kg}$ and $m_3 = 1.5 \text{ kg}$ respectively. Carts B and A are connected by a light taut inelastic string which passes over a light smooth pulley attaches to the cart C as shown. For this problem, all resistive and frictional forces may be ignored as may the moments of inertia of the pulley and of the wheels of all three carts. Take the acceleration due to gravity g to be 9.81 m s^{-2} .

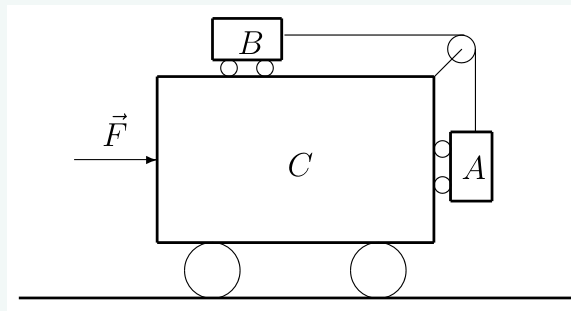


Figure 1:

1. A horizontal force $\vec{F}$ is now applied to cart C as shown. The size of $\vec{F}$ is such that carts A and B remain at rest relative to cart C .
 - a) Find the tension in the string connecting carts A and B .
 - b) Determine the magnitude of $\vec{F}$.
2. Later cart C is held stationary, while carts A and B are released from rest.
 - a) Determine the accelerations of carts A and B .
 - b) Calculate also the tension in the string.

Official English text of the problem.

Answers

- 1(a): $T = 2.94 \text{ N}$.
- 1(b): $F = 29.4 \text{ N}$.

- **2(a):** Cart A accelerates **downwards** at 5.89 m s^{-2} ; cart B accelerates **to the right, towards the pulley**, at 5.89 m s^{-2} .
- **2(b):** $T = 1.18 \text{ N}$.

Solution

The light string and ideal pulley give the same tension T throughout the string. With friction neglected, each contact force is normal to its supporting surface. We apply Newton's second law in the ground frame.

1(a)

Since A remains at rest relative to C , and C moves horizontally, A has no vertical acceleration. Its vertical forces are its weight m_1g downwards and the tension T upwards. Thus

$$T - m_1g = 0,$$

giving

$$T = m_1g = 0.3(9.81) = 2.943 \text{ N} \simeq 2.94 \text{ N}.$$

The horizontal normal force from C supplies A 's horizontal acceleration.

1(b)

Let a_C be the common horizontal acceleration of all three carts. The only horizontal force on B is the string tension, so

$$T = m_2a_C.$$

Consequently,

$$a_C = \frac{m_1g}{m_2} = \frac{0.3(9.81)}{0.2} = 14.715 \text{ m s}^{-2}.$$

For the complete system of carts, string and pulley, the contact forces and string forces are internal. The only external horizontal force is F . Therefore,

$$F = (m_1 + m_2 + m_3)a_C = \frac{m_1(m_1 + m_2 + m_3)}{m_2}g.$$

Numerically,

$$F = (0.3 + 0.2 + 1.5)(14.715) = 29.43 \text{ N} \simeq 29.4 \text{ N}.$$

2(a)

With C held stationary, the pulley is fixed. If A moves downwards by a distance s , the vertical string segment lengthens by s . Since the string is taut and inelastic, the horizontal segment must shorten by the same amount: B moves towards the pulley by s . Their accelerations therefore have equal magnitudes.

Let this magnitude be a , taking downwards as positive for A and rightwards as positive for B . Newton's second law gives

$$m_1g - T = m_1a$$

for A , and

$$T = m_2 a$$

for B . Adding eliminates the tension:

$$m_1 g = (m_1 + m_2) a.$$

Hence

$$a = \frac{m_1 g}{m_1 + m_2} = \frac{0.3(9.81)}{0.3 + 0.2} = 5.886 \text{ m s}^{-2} \simeq 5.89 \text{ m s}^{-2}.$$

Thus A accelerates downwards and B accelerates horizontally to the right, towards the pulley.

2(b)

Using the horizontal equation for B ,

$$T = m_2 a = \frac{m_1 m_2 g}{m_1 + m_2}.$$

Therefore,

$$T = 0.2(5.886) = 1.1772 \text{ N} \simeq 1.18 \text{ N}.$$

The solution was checked against the official solution.

Problem 2: Ice dropped into water in a copper calorimeter

Water of mass m_2 is contained in a copper calorimeter of mass m_1 . Their common temperature is t_2 . A piece of ice of mass m_3 and temperature $t_3 < 0^\circ\text{C}$ is dropped into the calorimeter.

a) Determine the temperature and masses of water and ice in the equilibrium state for general values of m_1, m_2, m_3, t_2 and t_3 . Write equilibrium equations for all possible processes which have to be considered.

b) Find the final temperature and final masses of water and ice for $m_1 = 1.00 \text{ kg}$, $m_2 = 1.00 \text{ kg}$, $m_3 = 2.00 \text{ kg}$, $t_2 = 10^\circ\text{C}$, $t_3 = -20^\circ\text{C}$.

Neglect the energy losses, assume the normal barometric pressure. Specific heat of copper is $c_1 = 0.1 \text{ kcal/kg} \cdot ^\circ\text{C}$, specific heat of water $c_2 = 1 \text{ kcal/kg} \cdot ^\circ\text{C}$, specific heat of ice $c_3 = 0.492 \text{ kcal/kg} \cdot ^\circ\text{C}$, latent heat of fusion of ice $l = 78.7 \text{ kcal/kg}$. Take $1 \text{ cal} = 4.2 \text{ J}$.

Official English text of the problem.

Answers

- a) Define the total water-plus-ice mass and the heat surplus at 0°C :

$$M = m_2 + m_3, \quad D = (m_1 c_1 + m_2 c_2) t_2 + m_3 c_3 t_3.$$

The equilibrium temperature t_f and masses are:

Condition	t_f	Final liquid-water mass	Final ice mass
$D < -m_2 l$	$\frac{D + m_2 l}{m_1 c_1 + M c_3} < 0$	0	M

Condition	t_f	Final liquid-water mass	Final ice mass
$-m_2l \leq D \leq m_3l$	0°C	$m_2 + \frac{D}{l}$	$m_3 - \frac{D}{l}$
$D > m_3l$	$\frac{D - m_3l}{m_1c_1 + Mc_2} > 0$	M	0

In the middle regime, $D < 0$ causes freezing, $D > 0$ causes melting, and $D = 0$ leaves the phase masses unchanged. The endpoints correspond to all ice or all water at 0°C .

• b)

$$t_f = 0^\circ\text{C}, \quad m_{\text{water}} = 0.890 \text{ kg}, \quad m_{\text{ice}} = 2.11 \text{ kg}.$$

Approximately 0.110 kg of the original water freezes.

Solution

a)

With no heat loss, the heat released by cooling and freezing equals the heat absorbed by warming and melting. We use the supplied specific heats and latent heat as constants. At normal barometric pressure, liquid water and ice coexist in equilibrium at 0°C .

Let

$$A = (m_1c_1 + m_2c_2)t_2$$

be the heat released when the calorimeter and the original water cool to 0°C , and let

$$B = m_3c_3(0 - t_3)$$

be the heat required to warm the original ice to 0°C . Their difference is

$$D = A - B = (m_1c_1 + m_2c_2)t_2 + m_3c_3t_3.$$

The possible processes follow from comparing this difference with the latent heats available.

Partial melting, with final temperature 0°C .

If $0 \leq D \leq m_3l$, the sensible heat surplus melts a mass x of ice. The equilibrium equation is

$$(m_1c_1 + m_2c_2)t_2 = m_3c_3(0 - t_3) + xl,$$

or $A = B + xl$. Thus

$$x = \frac{D}{l}, \quad t_f = 0^\circ\text{C},$$

$$m_{\text{water}} = m_2 + \frac{D}{l}, \quad m_{\text{ice}} = m_3 - \frac{D}{l}.$$

At $D = 0$, no net phase change occurs. At $D = m_3l$, the last ice melts and the final state is entirely water at 0°C .

Complete melting, with final temperature above 0°C .

If $D > m_3l$, all the ice melts. The calorimeter and original water cool from t_2 to t_f , while the added ice warms to zero, melts, and then warms as liquid water to t_f :

$$(m_1c_1 + m_2c_2)(t_2 - t_f) = m_3c_3(0 - t_3) + m_3l + m_3c_2t_f.$$

Collecting the terms containing t_f gives

$$D - m_3 l = [m_1 c_1 + (m_2 + m_3) c_2] t_f.$$

Therefore

$$t_f = \frac{D - m_3 l}{m_1 c_1 + (m_2 + m_3) c_2}, \quad m_{\text{water}} = m_2 + m_3, \quad m_{\text{ice}} = 0.$$

Partial freezing, with final temperature 0°C .

If $-m_2 l \leq D < 0$, cooling the calorimeter and water to zero supplies insufficient heat to warm the original ice to zero. A mass y of the original water must freeze, releasing latent heat:

$$(m_1 c_1 + m_2 c_2) t_2 + y l = m_3 c_3 (0 - t_3).$$

Hence

$$y = -\frac{D}{l}, \quad t_f = 0^\circ\text{C},$$

$$m_{\text{water}} = m_2 - y = m_2 + \frac{D}{l}, \quad m_{\text{ice}} = m_3 + y = m_3 - \frac{D}{l}.$$

At $D = -m_2 l$, all the original water freezes, leaving only ice at 0°C .

Complete freezing, with final temperature below 0°C .

If $D < -m_2 l$, even freezing all the original water cannot bring the system to zero. The final state contains only ice at $t_f < 0$.

The calorimeter cools from t_2 to t_f . The original water cools to zero, freezes, and then cools as ice to t_f . Their released heat warms the original ice from t_3 to t_f :

$$m_1 c_1 (t_2 - t_f) + m_2 c_2 t_2 + m_2 l + m_2 c_3 (0 - t_f) = m_3 c_3 (t_f - t_3).$$

Rearranging,

$$D + m_2 l = [m_1 c_1 + (m_2 + m_3) c_3] t_f,$$

so

$$t_f = \frac{D + m_2 l}{m_1 c_1 + (m_2 + m_3) c_3}, \quad m_{\text{water}} = 0, \quad m_{\text{ice}} = m_2 + m_3.$$

These cases exhaust the possible equilibrium states.

b)

The calorimeter and water can release

$$A = (1.00 \times 0.1 + 1.00 \times 1) \times 10 = 11.0 \text{ kcal}$$

by cooling to 0°C .

The heat needed to warm the original ice to zero is

$$B = 2.00 \times 0.492 \times 20 = 19.68 \text{ kcal}.$$

Consequently,

$$D = A - B = -8.68 \text{ kcal}.$$

Since

$$-m_2 l = -78.7 \text{ kcal} < D < 0,$$

only part of the original water freezes, and the equilibrium temperature is 0°C .

The mass that freezes is

$$y = \frac{B - A}{l} = \frac{8.68 \text{ kcal}}{78.7 \text{ kcal/kg}} = 0.110292 \dots \text{ kg}.$$

Thus

$$\boxed{m_{\text{water}} = 1.00 - y = 0.890 \text{ kg}},$$
$$\boxed{m_{\text{ice}} = 2.00 + y = 2.11 \text{ kg}}, \quad \boxed{t_f = 0^\circ\text{C}}.$$

The solution was checked against the official solution.

Problem 3: Charged ball hanging on the axis of a charged ring

A small charged ball of mass m and charge q is suspended from the highest point of a ring of radius R by means of an insulating cord of negligible mass. The ring is made of a rigid wire of negligible cross section and lies in a vertical plane. On the ring there is uniformly distributed charge Q of the same sign as q . Determine the length l of the cord so as the equilibrium position of the ball lies on the symmetry axis perpendicular to the plane of the ring.

Find first the general solution and then for particular values $Q = q = 9.0 \cdot 10^{-8} \text{ C}$, $R = 5 \text{ cm}$, $m = 1.0 \text{ g}$, $\epsilon_0 = 8.9 \cdot 10^{-12} \text{ F/m}$.

Official English text of the problem.

Answers

- 3. For equilibrium on the axis **away from the ring's centre**,

$$\boxed{l = \left(\frac{QqR}{4\pi\epsilon_0 mg} \right)^{1/3}},$$

provided

$$\frac{Qq}{4\pi\epsilon_0} > mgR^2.$$

The two possible positions are at axial distances $x = \pm\sqrt{l^2 - R^2}$. For the given data,

$$\boxed{l \approx 7.2 \text{ cm}}.$$

There is also the special central equilibrium $l = R$, which always satisfies static force balance. For these data, it has $l = 5 \text{ cm}$ and is unstable.

Solution

3. Treat the small ball as a point charge, take gravity as uniform, and use the prescribed uniform charge distribution on the ring. The massless cord exerts tension along its length.

Let O be the ring's centre, A its highest point, and P the ball's position on the perpendicular axis. First consider $OP = x > 0$. Since $OA = R$ is perpendicular to OP ,

$$l^2 = R^2 + x^2.$$

By symmetry, the components of the electric field parallel to the ring's plane cancel. Every charge element dQ is a distance $\sqrt{R^2 + x^2} = l$ from the ball. Coulomb's law therefore gives the resultant electric force along the axis:

$$F_e = \int \frac{q dQ}{4\pi\epsilon_0 l^2} \frac{x}{l} = \frac{Qq x}{4\pi\epsilon_0 l^3}.$$

Because $Qq > 0$, this force points away from the ring.

The tension T points from P towards A . Its upward component is TR/l , and its component towards the ring's plane is Tx/l . Vertical and horizontal equilibrium require

$$\frac{TR}{l} = mg, \quad \frac{Tx}{l} = \frac{Qq x}{4\pi\epsilon_0 l^3}.$$

The first equation gives $T/l = mg/R$. Substituting into the second and cancelling $x \neq 0$,

$$\frac{mg}{R} = \frac{Qq}{4\pi\epsilon_0 l^3},$$

hence

$$l = \left(\frac{QqR}{4\pi\epsilon_0 mg} \right)^{1/3}.$$

A nonzero axial displacement requires $l > R$, so this solution exists only if

$$\frac{Qq}{4\pi\epsilon_0} > mgR^2.$$

When this holds, reflection symmetry gives two equilibrium positions, one on each side of the ring:

$$x = \pm \sqrt{\left(\frac{QqR}{4\pi\epsilon_0 mg} \right)^{2/3} - R^2}.$$

For the numerical calculation, use $g = 9.8 \text{ m s}^{-2}$, $R = 0.050 \text{ m}$, and $m = 1.0 \times 10^{-3} \text{ kg}$. Then

$$\frac{Qq}{4\pi\epsilon_0 mgR^2} \approx 2.96 > 1,$$

so the required noncentral equilibrium exists. Its cord length is

$$l = \left[\frac{(9.0 \times 10^{-8})^2 (0.050)}{4\pi(8.9 \times 10^{-12})(1.0 \times 10^{-3})(9.8)} \right]^{1/3} = 0.07176 \text{ m}.$$

Thus

$$l \approx 7.2 \text{ cm}.$$

The corresponding distance from the ring's centre is approximately 5.1 cm.

Finally, the cancellation of x excludes a separate case. At $x = 0$, the electric field vanishes by symmetry, and $l = R$, $T = mg$ give a central equilibrium for any charge magnitude. To check this case for the supplied data, displace the ball slightly perpendicular to the ring. To first order, the electric force

is $Qq x / (4\pi\epsilon_0 R^3)$, while the restoring horizontal tension is mgx/R . Therefore,

$$F_x \simeq \left(\frac{Qq}{4\pi\epsilon_0 R^3} - \frac{mg}{R} \right) x.$$

The coefficient is positive for the given data, so this displacement grows: the central equilibrium is unstable.

The solution was checked against the official solution.

Problem 4: Interference in an air layer between a glass plate and a cube

A glass plate is placed above a glass cube of 2 cm edges in such a way that there remains a thin air layer between them, see Figure 5.

Electromagnetic radiation of wavelength between 400 nm and 1150 nm (for which the plate is penetrable) incident perpendicular to the plate from above is reflected from both air surfaces and interferes. In this range only two wavelengths give maximum reinforcements, one of them is $\lambda = 400$ nm. Find the second wavelength. Determine how it is necessary to warm up the cube so as it would touch the plate. The coefficient of linear thermal expansion is $\alpha = 8.0 \cdot 10^{-6} \text{ }^\circ\text{C}^{-1}$, the refractive index of the air $n = 1$. The distance of the bottom of the cube from the plate does not change during warming up.

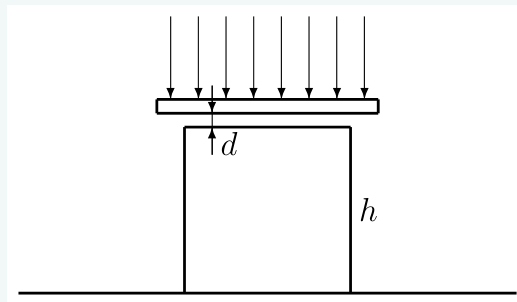


Figure 5:

Official English text of the problem.

Answers

- **Second wavelength:** $\lambda_2 \approx 667$ nm.
- **Required warming:** increase the cube's temperature by $\Delta T \approx 3.1$ °C. The initial air gap is $d = 500$ nm.

Solution

Second wavelength. Let d be the thickness of the air layer. At its upper boundary, reflection occurs from glass towards air and introduces no phase reversal. At its lower boundary, reflection occurs from air towards glass and introduces a phase reversal of π .

At normal incidence, the additional optical path travelled by the second reflected ray is $2nd$. Thus the phase difference is

$$\Delta\phi = \frac{4\pi nd}{\lambda} + \pi.$$

Constructive interference requires an integer multiple of 2π , giving

$$2nd = \left(m + \frac{1}{2}\right)\lambda, \quad m = 0, 1, 2, \dots$$

With $n = 1$, the wavelengths of reflection maxima are therefore

$$\lambda_m = \frac{4d}{2m + 1}.$$

Let the maximum at 400 nm have order m . Then

$$4d = (2m + 1)(400 \text{ nm}).$$

Larger orders give wavelengths below 400 nm, so the second maximum in the specified interval must have order $m - 1$. Consequently $m \geq 1$, and

$$\lambda_{m-1} = 400 \frac{2m + 1}{2m - 1} \text{ nm} \leq 1150 \text{ nm}.$$

Rearrangement gives

$$m \geq \frac{31}{30},$$

so $m \geq 2$.

There must be no third maximum in the interval. The next longer wavelength, corresponding to order $m - 2$, must therefore satisfy

$$\lambda_{m-2} = 400 \frac{2m + 1}{2m - 3} \text{ nm} > 1150 \text{ nm}.$$

This gives

$$m < \frac{77}{30}.$$

The only integer satisfying both bounds is $m = 2$. Hence

$$d = \frac{5(400 \text{ nm})}{4} = \boxed{500 \text{ nm}},$$

and the second wavelength is

$$\lambda_2 = \frac{4d}{3} = \frac{2000}{3} \text{ nm} \approx 667 \text{ nm}.$$

As a check, the next longer maximum is at 2000 nm, while the next shorter one is at $2000/7 \approx 286 \text{ nm}$; both lie outside the given interval.

Required warming. The cube initially has height

$$h = 2 \text{ cm} = 2.0 \times 10^{-2} \text{ m}.$$

Because the distance from its bottom to the plate remains fixed, contact occurs when its height increases by d .

Using linear thermal expansion with the given constant coefficient,

$$\Delta h = \alpha h \Delta T = d,$$

so

$$\Delta T = \frac{d}{\alpha h} = \frac{5.0 \times 10^{-7}}{(8.0 \times 10^{-6})(2.0 \times 10^{-2})} \text{ }^{\circ}\text{C} = 3.125 \text{ }^{\circ}\text{C}.$$

Thus the required temperature increase is

$$\boxed{\Delta T \approx 3.1 \text{ }^{\circ}\text{C} \quad (\text{equivalently, } 3.1 \text{ K})}.$$

The relative expansion is only $d/h = 2.5 \times 10^{-5}$, consistent with the linear approximation.

The solution was checked against the official solution.