

IPhO 1968

Theory problems and solutions · Budapest, Hungary · 3 theory problems with complete solutions

eulersolve.org/ipro/1968/

Problem 1: Solid cylinder and block joined by a cord on an inclined plane

On an inclined plane of 30° a block, mass $m_2 = 4$ kg, is joined by a light cord to a solid cylinder, mass $m_1 = 8$ kg, radius $r = 5$ cm (Fig. 1). Find the acceleration if the bodies are released. The coefficient of friction between the block and the inclined plane $\mu = 0.2$. Friction at the bearing and rolling friction are negligible.

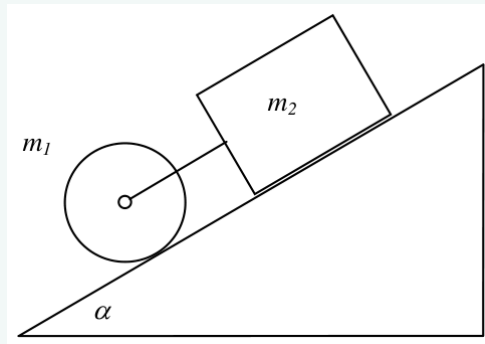


Figure 1

Official English text of the problem.

Answers

- **Problem 1:** Both bodies accelerate **down the inclined plane** with

$$a = \frac{(m_1 + m_2)g \sin 30^\circ - \mu m_2 g \cos 30^\circ}{\frac{3}{2}m_1 + m_2} = 0.3317g \approx 3.25 \text{ m s}^{-2}.$$

The numerical value uses $g = 9.81 \text{ m s}^{-2}$.

Solution

Problem 1

Treat the cord as massless and inextensible, and assume the cylinder rolls without slipping. With the cord taut, both bodies have the same acceleration a . Take the direction down the plane as positive and write $\theta = 30^\circ$.

The cord acts through the cylinder's central bearing, so its tension produces no torque about the cylinder's axis. Negligible rolling friction means that there is no rolling-resistance torque; static friction at the contact still supplies the torque needed for rotation.

Let T be the tension and f the static friction on the cylinder, directed up the plane.

For the block, the cord pulls down the plane. Its normal reaction and sliding friction are

$$N_2 = m_2 g \cos \theta, \quad F_2 = \mu m_2 g \cos \theta.$$

Newton's second law along the plane therefore gives

$$m_2 a = m_2 g \sin \theta + T - \mu m_2 g \cos \theta. \quad (1)$$

For the cylinder, both tension and static friction act up the plane:

$$m_1 a = m_1 g \sin \theta - T - f. \quad (2)$$

The moment of inertia of a solid cylinder about its axis is

$$I = \frac{1}{2} m_1 r^2.$$

If its angular acceleration has magnitude β , rolling without slipping requires $a = r\beta$. The rotational equation about its centre is consequently

$$f r = I \beta = \frac{1}{2} m_1 r^2 \frac{a}{r},$$

and hence

$$f = \frac{1}{2} m_1 a. \quad (3)$$

Adding (1) and (2), then substituting (3), eliminates the tension:

$$(m_1 + m_2) a = (m_1 + m_2) g \sin \theta - \mu m_2 g \cos \theta - \frac{1}{2} m_1 a.$$

Thus

$$a = \frac{(m_1 + m_2) g \sin \theta - \mu m_2 g \cos \theta}{\frac{3}{2} m_1 + m_2}.$$

The radius cancels because $I/r^2 = m_1/2$.

Substitution gives

$$a = g \frac{(8 + 4)(1/2) - (0.2)(4)(\sqrt{3}/2)}{(3/2)(8) + 4} = g \frac{5.30718}{16} = 0.331699g,$$

so

$$a \approx 3.25 \text{ m s}^{-2} \quad \text{down the plane.}$$

Finally, the assumed taut cord must require a positive tension. From (1),

$$T = m_2 a - m_2 g \sin \theta + \mu m_2 g \cos \theta \approx 0.192 \text{ N} > 0.$$

Thus the common-acceleration solution is consistent with a cord that can pull but cannot push.

The solution was checked against the official solution.

Problem 2: Mixing two quantities of toluene at different temperatures

There are 300 cm^3 toluene of 0°C temperature in a glass and 110 cm^3 toluene of 100°C temperature in another glass. (The sum of the volumes is 410 cm^3 .) Find the final volume after the two liquids are mixed. The coefficient of volume expansion of toluene $\beta = 0.001(^{\circ}\text{C})^{-1}$. Neglect the loss of heat.

Official English text of the problem.

Answers

- **Problem 2:** The final volume is $\boxed{410 \text{ cm}^3}$.

Solution

Use the standard linear thermal-expansion law for a fixed mass,

$$V(t) = V(0)(1 + \beta t),$$

where t is measured relative to 0°C . Assume that toluene's specific heat capacity is constant and that the heat capacities of the glasses are negligible.

First, determine the **mass ratio**, since heat exchange depends on mass. The hot portion's volume at 0°C would be

$$V_h(0) = \frac{110 \text{ cm}^3}{1 + (0.001 ^\circ\text{C}^{-1})(100^\circ\text{C})} = 100 \text{ cm}^3.$$

At the same temperature both portions have the same density, so

$$\frac{m_c}{m_h} = \frac{300}{100} = 3.$$

Conservation of energy requires the heat gained by the cold portion to equal the heat lost by the hot portion:

$$m_c c (t_f - 0^\circ\text{C}) = m_h c (100^\circ\text{C} - t_f).$$

Cancelling the common specific heat c and using the mass ratio gives

$$3t_f = 100^\circ\text{C} - t_f, \quad t_f = 25^\circ\text{C}.$$

The combined liquid would occupy $300 + 100 = 400 \text{ cm}^3$ at 0°C . Hence its volume at the final temperature is

$$V_f = (400 \text{ cm}^3) [1 + (0.001 ^\circ\text{C}^{-1})(25^\circ\text{C})] = \boxed{410 \text{ cm}^3}.$$

Equivalently, the cold portion expands to 307.5 cm^3 , while the hot portion contracts to 102.5 cm^3 . Their volume changes cancel within the linear-expansion model.

The solution was checked against the official solution.

Problem 3: Parallel rays refracted by a glass semi-cylinder

Parallel light rays are falling on the plane surface of a semi-cylinder made of glass, at an angle of 45° , in such a plane which is perpendicular to the axis of the semi-cylinder (Fig. 4). (Index of refraction is $\sqrt{2}$.) Where are the rays emerging out of the cylindrical surface?

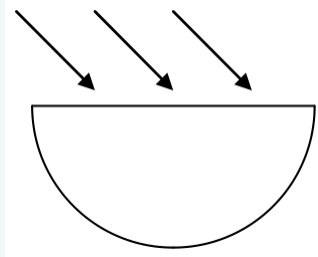


Figure 4

Official English text of the problem.

Answers

- **Problem 3.** Measure the central angle ϕ clockwise from the right-hand end of the horizontal diameter. The rays emerge through the arc bounded by

$$\phi = 15^\circ \quad \text{and} \quad \phi = 105^\circ.$$

This arc subtends 90° and has length $\pi R/2$, where R is the cylinder's radius. The limiting rays emerge tangentially.

Solution

Problem 3. We use geometrical optics, take the surrounding air's refractive index as 1, and neglect partial reflections when transmission is possible. Total internal reflection must, however, be included.

At the flat surface, Snell's law gives the refracted angle r , measured from the normal:

$$\sin 45^\circ = \sqrt{2} \sin r.$$

Hence

$$r = 30^\circ.$$

The rays inside the glass therefore travel downwards and to the right, at 60° below the horizontal.

Let O be the centre of the semicircular section. At a point P on the curved surface, its outward normal is the radius OP . Define ϕ as the angle of this radius measured clockwise from the horizontal direction to the right. For a ray reaching this surface directly, its incidence angle is

$$i = |60^\circ - \phi|.$$

The critical angle i_c for passage from glass to air satisfies

$$\sin i_c = \frac{1}{\sqrt{2}}, \quad i_c = 45^\circ.$$

Consequently, emergence is possible when

$$|60^\circ - \phi| \leq 45^\circ,$$

or

$$\boxed{15^\circ \leq \phi \leq 105^\circ}.$$

Thus the emitting arc begins 15° below the right-hand end of the diameter and ends 15° past the lowest point, towards the left. Its angular extent and length are

$$\Delta\phi = 90^\circ = \frac{\pi}{2}, \quad \boxed{s = \frac{\pi R}{2}}.$$

At either boundary, Snell's law gives an emergence angle of 90° relative to the normal, so the limiting outgoing rays are tangent to the circle. At $\phi = 60^\circ$, incidence is normal and the ray emerges without changing direction.

We must also check rays that initially undergo total internal reflection. Let a be a ray's entry coordinate along the diameter, measured from O , positive to the right. Its perpendicular distance from O is

$$b = |a| \cos 30^\circ = \frac{\sqrt{3}}{2} |a|.$$

At a circular-surface encounter,

$$b = R \sin i.$$

Specular reflection at a circle preserves this distance and hence the incidence angle at subsequent circular encounters. Any reflection at the flat diameter also preserves b , because that diameter passes through O . Therefore a ray initially totally reflected at the curved surface cannot subsequently escape through another part of that surface.

The rays contributing to the emergent arc are precisely those entering within

$$\frac{\sqrt{3}}{2} |a| \leq \frac{R}{\sqrt{2}}, \quad \boxed{|a| \leq R \sqrt{\frac{2}{3}} \approx 0.816R}.$$

The remaining incident rays add no further emitting region on the curved surface.

The solution was checked against the official solution.